

16. THE ENVELOPE THEOREM

J.P. Bastos*

EPH 1100: Analytical Tools

Reading: Hoy et al., §14.2–14.3.

- **§14.2:** The Slutsky equation.
- **§14.3:** The envelope theorem; the multiplier as the value of relaxing a constraint; derivatives of the expenditure function.

The book calls Hicksian demand *compensated* demand and Marshallian demand *uncompensated* demand.

1. Introduction

The price of gasoline rises by 10 cents a gallon. A household buys 600 gallons a year. How much worse off is it? Put in dollars: how much extra income would it need to be exactly as well off as before?

The quick answer is 600 gallons times 10 cents, or \$60. But the household does not keep buying 600 gallons. It drives a little less, carpools, combines errands. Each of these adjustments saves it some money, so \$60 seems to overstate the loss. To get the right answer, it seems we would need to know how the household re-optimizes: how much less it drives, and what it buys instead.

For a small price change, we do not. The household was already at its best choice, and near the best choice, small adjustments barely change how well off it is. So the adjustments matter little, and the quick answer is almost exactly right. This is the **envelope theorem**: to find how a change in a parameter (a price, income, a wage) affects the best a decision maker can do, hold the decision fixed and compute the direct effect of the parameter. Without a constraint, that is the effect on the objective. With a constraint, the parameter may appear only in the constraint, and the direct effect has to include it; Section 3 shows how.

We first see the theorem in a problem without a constraint, then with one. Section 4 applies it to the indirect utility function and Roy's identity; to the expenditure function and Shephard's lemma, which answers the gasoline question; to the firm's cost function; and to the Slutsky equation, which splits the response of demand to a price into a substitution effect and an income effect.

* Assistant Professor of Economics, University of Austin (UTX). jbastos@uauustin.org

2. The Envelope Theorem without a Constraint

A competitive firm sells its output at price p and has cost

$$C(q) = q^2$$

Its profit is

$$\pi(q, p) = pq - q^2$$

Profit depends on two things: the output the firm chooses and the price it faces. The firm chooses output to maximize profit. The first-order condition is

$$\frac{\partial \pi}{\partial q} = p - 2q = 0 \quad \implies \quad q^* = \frac{p}{2}$$

Substituting the best output into profit gives the most the firm can earn at each price, the **profit function**:

$$\pi^*(p) = p \cdot \frac{p}{2} - \left(\frac{p}{2}\right)^2 = \frac{p^2}{2} - \frac{p^2}{4} = \frac{p^2}{4}$$

A function like this one, the best value of the objective at each value of a parameter, is called a **value function**.

At a price of \$10, the firm produces 5 units and earns \$25. Suppose the price rises to \$11. We can compare two firms.

- A firm that keeps producing 5 units earns an extra dollar on each of its 5 units:

$$\pi(5, 11) = 55 - 25 = 30 \quad \text{gain} = 30 - 25 = 5$$

- A firm that re-optimizes produces 5.5 units:

$$\pi^*(11) = \frac{11^2}{4} = 30.25 \quad \text{gain} = 30.25 - 25 = 5.25$$

Re-optimizing adds only 25 cents to a gain of \$5. The firm was already producing the best output, so moving output a little away from it changes profit very little. Almost all of the gain comes from the higher price on the units the firm was already selling.

For a very small price change, the difference disappears. The derivative of the profit function is

$$\frac{d\pi^*}{dp} = \frac{2p}{4} = \frac{p}{2} = q^*$$

At a price of \$10 it is 5: each dollar of price raises maximum profit by about 5 dollars, the output the firm is producing. This is the same number we get by holding output fixed at 5 and taking the

partial derivative of profit with respect to the price:

$$\frac{\partial \pi}{\partial p} = q = 5$$

2.1 Why the Adjustment Does Not Matter

Maximum profit depends on the price in two ways: directly, and through the output the firm chooses. Write it as

$$\pi^*(p) = \pi(q^*(p), p)$$

and differentiate with the chain rule:

$$\frac{d\pi^*}{dp} = \underbrace{\frac{\partial \pi}{\partial p}}_{\text{direct effect}} + \underbrace{\frac{\partial \pi}{\partial q} \cdot \frac{dq^*}{dp}}_{\text{effect through output}}$$

The firm chooses output where the first-order condition holds, so at the best output

$$\frac{\partial \pi}{\partial q} = 0$$

and the second term is zero whatever the size of the output response. Only the direct effect is left.

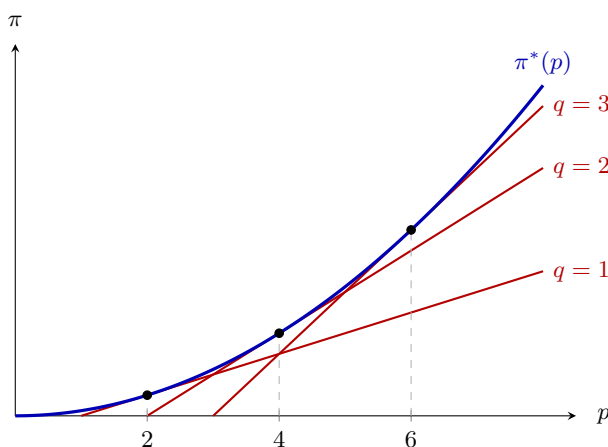


Figure 1: Maximum Profit as the Envelope of Fixed-Output Profit Lines.

Notes: Cost is $C(q) = q^2$. With output fixed, profit $pq - q^2$ is a straight line in the price with slope q . Maximum profit is $\pi^*(p) = p^2/4$. The line for each output touches the curve at the price where that output is the best one (price 2 for one unit, 4 for two units, 6 for three units), and there the slope of the curve equals the output.

Figure 1 shows the argument. Each straight line is the profit of a firm that holds its output fixed at 1, 2, or 3 units as the price changes: the quick answer, which ignores the adjustment. The curve is maximum profit, the firm that re-optimizes. At each price the best output gives the highest line, so the curve is the upper edge, or **envelope**, of the lines. Each line touches the curve at the price where its output is the best one, and there the two have the same slope. That shared slope is the theorem: the slope of maximum profit equals the slope of the fixed-output line, which

is the output. Near the touching point the line stays close to the curve, which is why re-optimizing added only 25 cents when the price rose from \$10 to \$11.

3. The Envelope Theorem with a Constraint

Suppose the objective, the constraint, or both depend on a parameter α :

$$\max_{x,y} f(x, y, \alpha) \quad \text{subject to} \quad g(x, y, \alpha) = c$$

To find the value function $V(\alpha)$, solve for the optimal choices as functions of the parameter and substitute them back into the objective.

Without a constraint, holding the choice fixed and differentiating the objective was enough. With a constraint it is not. In the consumer's problem below, utility does not contain the price of good 1 at all: the price appears only in the budget. Holding the bundle fixed and differentiating utility would say a price rise does not hurt the consumer, which is wrong. The price hurts her through the constraint: at the old bundle, she can no longer afford it. The Lagrangian contains both the objective and the constraint, so its partial derivative picks up the effect through the constraint. So with a constraint, the theorem applies to the Lagrangian.

Envelope Theorem. Write the Lagrangian

$$\mathcal{L}(x, y, \lambda, \alpha) = f(x, y, \alpha) + \lambda[c - g(x, y, \alpha)]$$

The derivative of the value function is the partial derivative of the Lagrangian with respect to the parameter, evaluated at the optimal choices and multiplier:

$$\frac{dV}{d\alpha} = \left. \frac{\partial \mathcal{L}}{\partial \alpha} \right|_{x^*, y^*, \lambda^*}$$

The indirect effects drop out for the same reason as before. At the optimum the constraint holds, so the bracket in the Lagrangian is zero and the Lagrangian equals the objective:

$$V(\alpha) = \mathcal{L}(x^*(\alpha), y^*(\alpha), \lambda^*(\alpha), \alpha)$$

The parameter moves the choices and the multiplier. Differentiate with the chain rule, one term for each:

$$\frac{dV}{d\alpha} = \underbrace{\frac{\partial \mathcal{L}}{\partial x} \cdot \frac{dx^*}{d\alpha}}_{\text{through } x} + \underbrace{\frac{\partial \mathcal{L}}{\partial y} \cdot \frac{dy^*}{d\alpha}}_{\text{through } y} + \underbrace{\frac{\partial \mathcal{L}}{\partial \lambda} \cdot \frac{d\lambda^*}{d\alpha}}_{\text{through } \lambda} + \underbrace{\frac{\partial \mathcal{L}}{\partial \alpha}}_{\text{direct effect}}$$

At the optimum, the first-order conditions make the first two partial derivatives zero, and the third

is the constraint itself:

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \quad \frac{\partial \mathcal{L}}{\partial y} = 0 \quad \frac{\partial \mathcal{L}}{\partial \lambda} = c - g(x^*, y^*, \alpha) = 0$$

All three indirect terms are zero, whatever the size of the responses of the choices and the multiplier. Only the direct effect is left.

Nothing in this argument uses the fact that the problem is a maximum. The first-order conditions and the constraint hold at a minimum too, so the theorem applies in the same way when a consumer minimizes spending or a firm minimizes cost.

4. Applications

4.1 The Indirect Utility Function

A consumer maximizes utility subject to her budget constraint:

$$\max_{x_1, x_2} U(x_1, x_2) \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 = m$$

Solving gives her Marshallian demands, which depend on prices and income. Substitute them back into the utility function. The result is her maximum utility at each set of prices and income, the **indirect utility function**:

$$V(p_1, p_2, m) = U(x_1^M(p_1, p_2, m), x_2^M(p_1, p_2, m))$$

The envelope theorem gives its derivatives. In the Lagrangian

$$\mathcal{L} = U(x_1, x_2) + \lambda(m - p_1 x_1 - p_2 x_2)$$

income and the price of good 1 appear only in the budget term, so

$$\frac{\partial V}{\partial m} = \frac{\partial \mathcal{L}}{\partial m} = \lambda^* \quad \frac{\partial V}{\partial p_1} = \frac{\partial \mathcal{L}}{\partial p_1} = -\lambda^* x_1^M$$

The first result says the multiplier is the marginal utility of income: the utility from one more dollar. Step 6 of the example below shows this with numbers.

The second result says how much a price rise hurts her. Read it in two pieces:

- If she kept buying the same bundle, a \$1 rise in the price of good 1 would cost her one extra dollar for each unit of good 1, or x_1^M dollars in total.
- Each dollar is worth λ^* units of utility, so the loss in utility is λ^* times x_1^M .

She does not keep buying the same bundle: she buys less of good 1 and more of good 2. The envelope theorem says this adjustment does not change the answer for a small price change. So a

\$1 rise in the price of good 1 hurts her about as much as losing x_1^M dollars of income. The derivative is negative, and its size is larger the more of the good she buys. A rise in the price of a good she does not buy does not hurt her at all.

Dividing the two results cancels the multiplier and leaves the quantity she buys:

Roy's Identity. The Marshallian demand for a good is minus the derivative of the indirect utility function with respect to its price, divided by the derivative with respect to income:

$$x_1^M = -\frac{\partial V / \partial p_1}{\partial V / \partial m}$$

So if we know the indirect utility function, we can recover the demands from it by differentiating.

Example: A consumer has utility

$$U(x_1, x_2) = x_1^{0.5} x_2^{0.5}$$

Her Marshallian demands are

$$x_1^M = \frac{0.5 m}{p_1} \quad x_2^M = \frac{0.5 m}{p_2}$$

How much does one more dollar of income raise her maximum utility, and how much does a higher price of good 1 lower it? Answer by solving directly, then check that the envelope theorem gives the same answer.

Step 1: Substitute the demands into the utility function.

$$V = \left(\frac{0.5 m}{p_1} \right)^{0.5} \left(\frac{0.5 m}{p_2} \right)^{0.5} = \frac{0.5 m}{(p_1 p_2)^{0.5}}$$

Step 2: The effect of income on her maximum utility is the derivative of the indirect utility function with respect to income; the effect of the price is the derivative with respect to the price:

$$\frac{\partial V}{\partial m} = \frac{0.5}{(p_1 p_2)^{0.5}} \quad \frac{\partial V}{\partial p_1} = -0.5 \times \frac{0.5 m}{p_1^{1.5} p_2^{0.5}}$$

Step 3: Evaluate at prices of \$2 for good 1 and \$4 for good 2 and an income of \$100.

$$\frac{\partial V}{\partial m} = \frac{0.5}{8^{0.5}} \approx 0.177 \quad \frac{\partial V}{\partial p_1} = -\frac{25}{2^{1.5} \times 2} = -\frac{25}{5.66} \approx -4.42$$

Step 4: Compare with the envelope theorem. The first-order condition for good 1 gives the multiplier:

$$0.5 x_1^{-0.5} x_2^{0.5} = \lambda p_1$$

She buys 25 units of good 1 and 12.5 units of good 2, so the multiplier is

$$\lambda^* = \frac{0.5 x_1^{-0.5} x_2^{0.5}}{p_1} = \frac{0.5 \times 25^{-0.5} \times 12.5^{0.5}}{2} = \frac{0.5 \times 0.2 \times 3.54}{2} \approx 0.177$$

The envelope theorem then gives

$$\frac{\partial V}{\partial m} = \lambda^* \approx 0.177 \quad \frac{\partial V}{\partial p_1} = -\lambda^* x_1^M \approx -0.177 \times 25 \approx -4.42$$

Both match Step 3. Here the envelope theorem is only a check, because we already had the indirect utility function. When that function is unknown, the envelope theorem gives the same answers from the multiplier and the bundle she buys alone.

Step 5: Check Roy's identity.

$$-\frac{\partial V / \partial p_1}{\partial V / \partial m} = -\frac{-4.42}{0.177} \approx 25 = x_1^M$$

Step 6: See the multiplier as a shadow price. Give her one more dollar of income:

$$\begin{aligned} V(2, 4, 101) &= \frac{0.5 \times 101}{8^{0.5}} = \frac{50.5}{2.8284} \approx 17.854 \\ V(2, 4, 100) &= \frac{0.5 \times 100}{8^{0.5}} = \frac{50}{2.8284} \approx 17.678 \end{aligned}$$

Utility rises by about 0.177, the multiplier. The last dollar of her budget is worth 0.177 units of utility. This is why the multiplier is called a shadow price: it is what one more unit of the constrained resource is worth to her, even though no market sets that price.

4.2 Shephard's Lemma

The **expenditure function** is the least it costs to reach a target utility $\bar{U}$ at given prices:

$$E(p_1, p_2, \bar{U}) = \min_{x_1, x_2} p_1 x_1 + p_2 x_2 \quad \text{subject to} \quad U(x_1, x_2) = \bar{U}$$

The cheapest bundle is the **Hicksian demand**, which depends on prices and the target utility. In the Lagrangian

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \mu(\bar{U} - U(x_1, x_2))$$

the price of good 1 appears only in the term for spending on good 1, so

$$\frac{\partial \mathcal{L}}{\partial p_1} = x_1$$

By the envelope theorem, evaluated at the Hicksian demand:

Shephard's Lemma. The derivative of the expenditure function with respect to the price of a good is the Hicksian demand for that good:

$$\frac{\partial E}{\partial p_1} = x_1^H(p_1, p_2, \bar{U}) \quad \frac{\partial E}{\partial p_2} = x_2^H(p_1, p_2, \bar{U})$$

So the extra income the consumer needs per dollar of price increase is the quantity she buys. For the household that buys 600 gallons, a 10-cent rise in the price of gasoline costs about \$60. The adjustments the household makes (driving less, buying other things) would lower the cost of staying as well off, but for a small price change they lower it very little.

Example: The consumer in Section 4.1 has Hicksian demands

$$x_1^H = \bar{U} \left(\frac{p_2}{p_1} \right)^{0.5} \quad x_2^H = \bar{U} \left(\frac{p_1}{p_2} \right)^{0.5}$$

When the price of good 1 rises, how much extra income does she need to stay as well off as before?

Step 1: Find the expenditure function. Substitute the Hicksian demands into spending:

$$E = p_1 \bar{U} \left(\frac{p_2}{p_1} \right)^{0.5} + p_2 \bar{U} \left(\frac{p_1}{p_2} \right)^{0.5} = \bar{U} (p_1 p_2)^{0.5} + \bar{U} (p_1 p_2)^{0.5} = 2 \bar{U} (p_1 p_2)^{0.5}$$

Step 2: The extra income she needs per dollar of price increase is the derivative of the expenditure function with respect to the price of good 1:

$$\frac{\partial E}{\partial p_1} = 2 \bar{U} \times 0.5 p_1^{-0.5} p_2^{0.5} = \bar{U} \left(\frac{p_2}{p_1} \right)^{0.5} = x_1^H$$

This is Shephard's lemma.

Step 3: Evaluate. Prices are \$2 for good 1 and \$4 for good 2, and her utility is 17.678, what she reaches with \$100 at these prices (Section 4.1). The price of good 1 rises by 10 cents.

$$\frac{\partial E}{\partial p_1} = 17.678 \times \left(\frac{4}{2} \right)^{0.5} = 17.678 \times 1.414 \approx 25$$

The derivative is the 25 units she buys. So she needs about

$$\Delta E \approx 25 \times 0.10 = 2.50$$

Step 4: Check with the expenditure function at the new price.

$$E(2.10, 4, 17.678) = 2 \times 17.678 \times (2.10 \times 4)^{0.5} = 35.356 \times 2.898 \approx 102.47$$

$$E(2, 4, 17.678) = 2 \times 17.678 \times (2 \times 4)^{0.5} = 35.356 \times 2.828 \approx 100.00$$

She needs \$2.47, three cents less than the prediction. The difference is what she saves by buying a little less of good 1 and a little more of good 2.

4.3 Production Theory

The same results hold for a firm. Its **minimum cost function** is the least it costs to produce $\bar{Q}$ units at wage w and rental rate r :

$$C^*(w, r, \bar{Q}) = \min_{K, L} wL + rK \quad \text{subject to} \quad F(K, L) = \bar{Q}$$

The cheapest inputs are the **conditional input demands**. In the Lagrangian

$$\mathcal{L} = wL + rK + \mu(\bar{Q} - F(K, L))$$

the wage appears only in the wage bill, the rental rate only in the cost of capital, and the required output only in the bracket:

$$\frac{\partial \mathcal{L}}{\partial w} = L \quad \frac{\partial \mathcal{L}}{\partial r} = K \quad \frac{\partial \mathcal{L}}{\partial \bar{Q}} = \mu$$

By the envelope theorem, evaluated at the cheapest inputs:

$$\frac{\partial C^*}{\partial w} = L^* \quad \frac{\partial C^*}{\partial r} = K^* \quad \frac{\partial C^*}{\partial \bar{Q}} = \mu^*$$

The first two are Shephard's lemma for the firm: a \$1 rise in the wage raises minimum cost by about the number of workers it hires. The third says the multiplier is marginal cost, the cost of producing one more unit.

Shephard's lemma is used in applied work in the other direction. Economists often estimate a firm's or an industry's cost function from data on costs and prices. Differentiating the estimated cost function then gives the input demands, without writing down a production function at all.

4.4 The Slutsky Equation

The price of good 1 rises, and the consumer buys less of it, for two reasons. First, good 1 is now more expensive compared with good 2, so she shifts toward good 2: the **substitution effect**. Second, her income now buys fewer goods than before, so she is poorer in real terms: the **income effect**. How much of the fall in demand comes from each?

The two demand functions separate the two effects. The Marshallian demand holds income fixed, so when the price rises she is both facing a new relative price and poorer. Its derivative with respect to the price is the **total effect**. The Hicksian demand holds utility fixed: as the price rises, she receives whatever income keeps her as well off as before, so she is not poorer, and any change in what she buys comes only from the change in relative prices. Its derivative is the substitution effect.

4.4.1 Deriving the Equation

The two problems give the same bundle when the consumer's income is exactly the expenditure needed to reach the target utility. That is, at any prices, the Hicksian demand equals the Marshallian demand evaluated at that income:

$$x_1^H(p_1, p_2, \bar{U}) = x_1^M(p_1, p_2, E(p_1, p_2, \bar{U}))$$

This holds at every price of good 1, so the derivatives of the two sides with respect to that price are equal.

Step 1: Differentiate both sides with respect to the price of good 1. On the right, the price enters the Marshallian demand directly, and also through the expenditure that is plugged in for income, so use the chain rule:

$$\frac{\partial x_1^H}{\partial p_1} = \frac{\partial x_1^M}{\partial p_1} + \frac{\partial x_1^M}{\partial m} \cdot \frac{\partial E}{\partial p_1}$$

Step 2: Replace the derivative of the expenditure function using Shephard's lemma. It is the Hicksian demand, which at the starting point is the quantity she buys, x_1 :

$$\frac{\partial x_1^H}{\partial p_1} = \frac{\partial x_1^M}{\partial p_1} + x_1 \cdot \frac{\partial x_1^M}{\partial m}$$

Step 3: Solve for the total effect.

The Slutsky Equation.

$$\underbrace{\frac{\partial x_1^M}{\partial p_1}}_{\text{total effect}} = \underbrace{\frac{\partial x_1^H}{\partial p_1}}_{\text{substitution effect}} - \underbrace{x_1 \cdot \frac{\partial x_1^M}{\partial m}}_{\text{income effect}}$$

The income effect is minus the response of demand to income, scaled by how much she buys. A \$1 rise in the price of a good she buys 25 units of makes her about \$25 poorer in real terms, which is the result of Shephard's lemma.

The envelope theorem is what makes this derivation short. Step 1 of the derivation needs to know how much income it takes to keep her as well off when the price rises. That seems to require knowing how her whole bundle changes, since she buys less of good 1 and more of good 2. The envelope theorem says that for a small price change, the adjustment does not matter: the extra income is the quantity she buys. Without it, we would have to differentiate all three first-order conditions of her problem at the same time and solve the resulting system of equations, which is how the textbook first derives the equation.

4.4.2 Signs

The substitution effect is always negative (or zero). With her utility held fixed, she moves along her indifference curve, and a higher relative price of good 1 moves her toward less of good 1 and more of good 2.

The sign of the income effect depends on the good.

- For a **normal good**, demand rises with income, so the income effect is negative. The two effects push in the same direction, and demand falls by more than the substitution effect alone.
- For an **inferior good**, demand falls with income, so the income effect is positive and works against the substitution effect, and demand falls by less.
- If the good is inferior and takes a large share of the budget, the income effect can be larger in size than the substitution effect, and demand rises with the price. Such a good is called a **Giffen good**. It is rare in practice, but the equation shows what it requires.

4.4.3 The Equation in Elasticities

Multiply both sides of the Slutsky equation by the price of good 1 over the quantity of good 1:

$$\frac{\partial x_1^M}{\partial p_1} \cdot \frac{p_1}{x_1} = \frac{\partial x_1^H}{\partial p_1} \cdot \frac{p_1}{x_1} - p_1 \cdot \frac{\partial x_1^M}{\partial m}$$

The left side is the own-price elasticity of Marshallian demand, ε_{11} . The first term on the right is the same elasticity for Hicksian demand, ε_{11}^H . Multiply and divide the last term by income over quantity, and it becomes the budget share times the income elasticity:

$$p_1 \cdot \frac{\partial x_1^M}{\partial m} = \frac{p_1 x_1}{m} \cdot \left(\frac{\partial x_1^M}{\partial m} \cdot \frac{m}{x_1} \right) = s_1 \eta_1$$

So

$$\varepsilon_{11} = \varepsilon_{11}^H - s_1 \eta_1$$

The income effect is large only when the good takes a large share of the budget and demand responds strongly to income. A rise in the price of salt barely makes anyone poorer, so almost all of the response is substitution. A rise in rents makes renters noticeably poorer, so the income effect matters.

Example: For the consumer in Section 4.1, split the response of her demand for good 1 to its price into the substitution and income effects. Her Marshallian demand is

$$x_1^M = \frac{0.5m}{p_1}$$

Step 1: Find the total effect, at an income of \$100 and a price of \$2 for good 1.

$$\frac{\partial x_1^M}{\partial p_1} = -\frac{0.5 m}{p_1^2} = -\frac{50}{4} = -12.5$$

Step 2: Find the income effect. At these values she buys 25 units of good 1.

$$\frac{\partial x_1^M}{\partial m} = \frac{0.5}{p_1} = 0.25 \quad -x_1 \cdot \frac{\partial x_1^M}{\partial m} = -25 \times 0.25 = -6.25$$

Step 3: Find the substitution effect from the Slutsky equation. Total effect equals substitution effect plus income effect:

$$\begin{aligned} -12.5 &= \frac{\partial x_1^H}{\partial p_1} - 6.25 \\ \frac{\partial x_1^H}{\partial p_1} &= -12.5 + 6.25 = -6.25 \end{aligned}$$

Of the 12.5 units she gives up per dollar of price increase, half is substitution and half is income.

Step 4: Check by differentiating the Hicksian demand directly, at a price of \$4 for good 2 and her utility of 17.678.

$$\frac{\partial x_1^H}{\partial p_1} = -0.5 \bar{U} p_2^{0.5} p_1^{-1.5} = -0.5 \times 17.678 \times \frac{2}{2.828} = -6.25$$

Step 5: Restate in elasticities. Her budget share for good 1 is one half, and the elasticities are

$$\varepsilon_{11} = -12.5 \times \frac{2}{25} = -1 \quad \eta_1 = 0.25 \times \frac{100}{25} = 1 \quad \varepsilon_{11}^H = -6.25 \times \frac{2}{25} = -0.5$$

and the elasticity form of the Slutsky equation holds:

$$-1 = -0.5 - 0.5 \times 1$$

A 1% rise in the price of good 1 cuts her demand by 1%: half a percent because good 1 became relatively expensive, and half a percent because she became poorer.

Practice Problems

1. A competitive firm has production function $q = L^{0.5}$, sells output at price p , and pays wage w .
 - (a) Write profit as a function of labor, and find the profit-maximizing labor L^* and output q^* .
 - (b) Substitute to find the profit function $\pi^*(p, w)$.
 - (c) Use the envelope theorem to find $\partial\pi^*/\partial p$ and $\partial\pi^*/\partial w$ without differentiating π^* . Then differentiate π^* directly and check.
 - (d) At $p = 10$ and $w = 1$, by about how much does maximum profit fall if the wage rises to \$1.10? Explain in words why the answer is the number of workers times the change in the wage.

2. A consumer has utility $U(x_1, x_2) = x_1^{0.25}x_2^{0.75}$, and her indirect utility function is

$$V(p_1, p_2, m) = \frac{A m}{p_1^{0.25} p_2^{0.75}}$$

where A is a positive constant.

- (a) Find $\partial V/\partial m$. What does the envelope theorem say it equals?
 - (b) Find $\partial V/\partial p_1$. Use the envelope theorem to explain its sign.
 - (c) Divide your answer to (b) by your answer to (a), and change the sign. Show that the result is her Marshallian demand for good 1.
3. A firm's minimum cost function is $C^*(w, r, \bar{Q}) = 3w^{1/3}r^{2/3}\bar{Q}$.
 - (a) Use Shephard's lemma to find the conditional labor demand $L^*(w, r, \bar{Q})$.
 - (b) Use Shephard's lemma to find the conditional capital demand $K^*(w, r, \bar{Q})$.
 - (c) Verify that $wL^* + rK^* = C^*$.
 - (d) At $w = 8$, $r = 1$, and $\bar{Q} = 10$, by about how much does minimum cost rise if the wage rises by \$0.50?
4. A household buys 600 gallons of gasoline a year at \$3.50 a gallon. The price rises to \$3.70.
 - (a) Use Shephard's lemma to estimate the extra income the household needs to be as well off as before.
 - (b) Is the exact amount larger or smaller than your estimate? Explain in words.
 - (c) A government plans to compensate every household with the amount in (a). Would households be better off, worse off, or as well off as before the price rise? Explain.

5. A consumer has utility $U(x_1, x_2) = x_1^{0.25}x_2^{0.75}$, income $m = 120$, and faces prices $p_1 = 1$ and $p_2 = 3$.
- (a) Find her Marshallian demand for good 1 and how much of it she buys.
 - (b) Find the total effect, the income effect, and the substitution effect of a change in the price of good 1.
 - (c) Restate the Slutsky equation in elasticities, and check that it holds with your numbers.
6. A consumer has utility $U(x_1, x_2) = 2 \ln x_1 + x_2$, and her income is large enough that she buys some of both goods.
- (a) Find her Marshallian demand for good 1. Does it depend on income?
 - (b) What is the income effect of a change in the price of good 1? What does the Slutsky equation say about the total and the substitution effects?
 - (c) Name a good for which this utility function might be a reasonable description, and explain why.
7. At current prices, a low-income household buys 50 kilograms of rice a month. Its Hicksian demand for rice falls by 2 kilograms for each \$1 rise in the price of rice, and its Marshallian demand for rice falls by 0.1 kilograms for each \$1 rise in income.
- (a) Is rice a normal or an inferior good for this household?
 - (b) Use the Slutsky equation to find the total effect of a \$1 rise in the price of rice on the quantity it buys.
 - (c) Interpret the result. What is a good of this kind called, and what features of rice and of the household make the result possible?
8. Explain in words, without equations, why the envelope theorem lets us hold the optimal choices fixed when we find how a parameter affects the value function. Use the firm in Section 2 as your example.