

# 15. COMPARATIVE STATICS AND THE IMPLICIT FUNCTION THEOREM

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EPH 1100: Analytical Tools

**Reading:** Hoy et al., §11.4, §14.1, and the first part of §14.2.

- **§11.4:** Implicit differentiation; the implicit function theorem (Theorems 11.2 and 11.3); slopes of level curves.
- **§14.1:** Comparative statics in two simple models (Example 14.1, a tax on a monopoly, is the one to read closely).
- **§14.2:** Read through Example 14.3 (one endogenous variable, general functions). The rest of the section, with several endogenous variables, is beyond this course.

## 1. Introduction

Economists spend much of their time on questions of the form “what happens to  $A$  when  $B$  changes?” If the government taxes a product, how much of the tax do buyers pay, and how much do sellers pay? If household income rises, how much does the price of a good go up? If the wage rises, how many fewer workers does a firm hire?

Questions like these are called **comparative statics**. The name comes from comparing two equilibria, one before the change and one after. We do not describe how the economy moves from one to the other, only where it ends up.

If we knew the exact demand and supply functions, these questions would be easy. We would solve for the equilibrium price without the tax, plug in the tax, solve again, and compare the two prices.

Usually we do not know the functions. We know the signs of their slopes: demand slopes down and supply slopes up. We may have an estimate of how steep each one is near the current equilibrium, but not a formula. The tools in this note let us say which direction the equilibrium moves, and by about how much, with only that information. What we always have is an *equation* the equilibrium must satisfy, such as demand equals supply. Differentiating both sides of that equation gives the derivative of the equilibrium with respect to the thing that changes, without

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ever solving for the equilibrium. The answer is written in terms of the slopes of demand and supply, so their signs alone tell us the direction of the change. The technique is called implicit differentiation, and the result that says when it works is the implicit function theorem.

We start with a tax in a market with specific numbers, where we can solve everything directly. Then we repeat the same analysis with general demand and supply functions, and check that plugging in the numbers gives the same answer.

## 2. A Tax with Known Demand and Supply

A town's restaurants sell meals. Daily demand and supply are

$$Q^D = 150 - 2P \qquad Q^S = 10 + 3P$$

where  $P$  is the price of a meal in dollars. Setting demand equal to supply gives the equilibrium:

$$150 - 2P = 10 + 3P$$

$$140 = 5P$$

$$P^* = 28$$

Plugging the price into demand gives the quantity:

$$Q^* = 150 - 2(28) = 94$$

The town now taxes restaurants  $t$  dollars per meal. The tax separates the price buyers (diners) pay,  $P_b$ , from the price sellers (restaurants) keep,  $P_s$ :

$$P_s = P_b - t$$

Diners buy based on what they pay, and restaurants supply based on what they keep:

$$Q^D = 150 - 2P_b \qquad Q^S = 10 + 3(P_b - t)$$

With a tax of \$10, demand equals supply when

$$150 - 2P_b = 10 + 3(P_b - 10)$$

$$150 - 2P_b = 3P_b - 20$$

$$170 = 5P_b$$

$$P_b^* = 34$$

Restaurants keep the price minus the tax, and the quantity comes from demand:

$$P_s^* = 34 - 10 = 24 \quad Q^* = 150 - 2(34) = 82$$

Diners pay \$6 more and restaurants keep \$4 less (Figure 1).

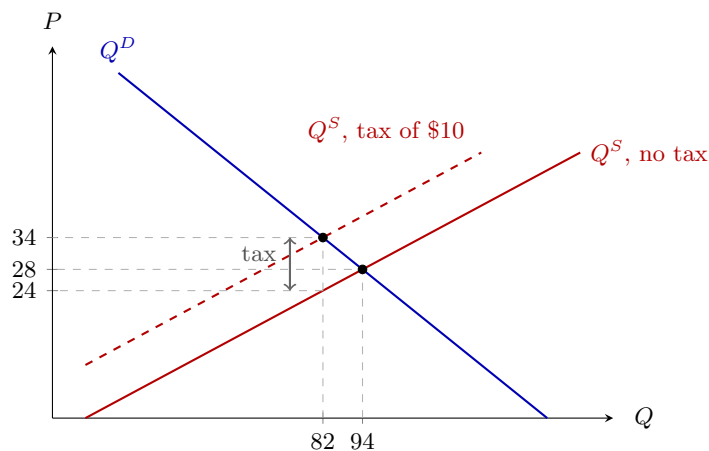


**Figure 1:** A Tax on Restaurant Meals.

*Notes:* Demand is  $Q^D = 150 - 2P_b$  and supply is  $Q^S = 10 + 3(P_b - t)$ , where  $P_b$  is the price diners pay. A tax of \$10 shifts the supply curve up by \$10: restaurants need \$10 more from diners to supply the same number of meals. The price diners pay rises from \$28 to \$34, and the price restaurants keep falls from \$28 to \$24.

Keeping the tax as a letter gives the price diners pay at any tax. Set demand equal to supply, expand the right side, and collect the price terms on one side:

$$150 - 2P_b = 10 + 3(P_b - t)$$

$$150 - 2P_b = 10 + 3P_b - 3t$$

$$140 + 3t = 5P_b$$

$$P_b^*(t) = 28 + 0.6t$$

The derivative with respect to the tax is the coefficient on the tax:

$$\frac{dP_b^*}{dt} = 0.6$$

Each dollar of tax raises the price diners pay by 60 cents. Diners pay 60% of the tax, and restaurants pay the other 40%.

### 3. Implicit Differentiation

Now suppose we do not know the formulas. All we know is that demand slopes down and supply slopes up:

$$Q^D(P_b) \quad \text{with} \quad \frac{dQ^D}{dP_b} < 0 \quad Q^S(P_b - t) \quad \text{with} \quad \frac{dQ^S}{dP_b} > 0$$

The tax is set by the town, outside the market: it is **exogenous**. The price is set by the market: it is **endogenous**. For each tax, the equilibrium price is the one at which demand equals supply, so it is a function of the tax,  $P_b^*(t)$ . We do not know its formula, but we know that at every tax it satisfies

$$Q^D(P_b^*(t)) = Q^S(P_b^*(t) - t)$$

Because this equation holds at every tax, the two sides change by the same amount when the tax changes. So we can differentiate both sides with respect to the tax.

Each side is a function of a function, so we use the chain rule: the slope of the outside function times the derivative of the inside. On the left, the inside is the price, and its derivative is the price response. On the right, the inside is the price minus the tax, and its derivative is the price response minus 1. This gives

$$\underbrace{\frac{dQ^D}{dP_b} \cdot \frac{dP_b^*}{dt}}_{\text{change in quantity demanded}} = \underbrace{\frac{dQ^S}{dP_b} \cdot \left(\frac{dP_b^*}{dt} - 1\right)}_{\text{change in quantity supplied}}$$

This is one equation in one unknown, the price response. Multiply out the right side, move both terms with the price response to the left, and factor it out:

$$\begin{aligned} \frac{dQ^D}{dP_b} \cdot \frac{dP_b^*}{dt} &= \frac{dQ^S}{dP_b} \cdot \frac{dP_b^*}{dt} - \frac{dQ^S}{dP_b} \\ \left(\frac{dQ^D}{dP_b} - \frac{dQ^S}{dP_b}\right) \cdot \frac{dP_b^*}{dt} &= -\frac{dQ^S}{dP_b} \end{aligned}$$

Divide both sides by the parentheses, and multiply the top and the bottom by  $-1$ :

$$\frac{dP_b^*}{dt} = \frac{dQ^S/dP_b}{dQ^S/dP_b - dQ^D/dP_b}$$

This is the share of the tax diners pay. Since the demand slope is negative, the bottom is the supply slope plus a positive number, so the share is between 0 and 1: diners pay part of the tax, but never all of it. Diners pay more than half of the tax when supply responds more to the price than demand does.

In the restaurant market, the supply slope is 3 and the demand slope is  $-2$ :

$$\frac{dP_b^*}{dt} = \frac{3}{3 - (-2)} = 0.6$$

the same answer as in Section 2.

### 3.1 Tax Incidence and Elasticities

The split of a tax between buyers and sellers is called its **incidence**. Slopes depend on units (meals per day, dollars). Elasticities do not, so the result is usually stated with them. Start before the tax, where the price diners pay and the price restaurants keep are the same,  $P^*$ , and the quantity is

$Q^*$ . Multiply the top and the bottom of the formula by price over quantity, which does not change its value:

$$\frac{dP_b^*}{dt} = \frac{\frac{dQ^S}{dP_b} \cdot \frac{P^*}{Q^*}}{\frac{dQ^S}{dP_b} \cdot \frac{P^*}{Q^*} - \frac{dQ^D}{dP_b} \cdot \frac{P^*}{Q^*}} = \frac{\varepsilon_S}{\varepsilon_S - \varepsilon_D}$$

where  $\varepsilon_S$  and  $\varepsilon_D$  are the price elasticities of supply and demand. Buyers' share is the supply elasticity over the supply elasticity plus the size of the demand elasticity. The less elastic side of the market pays more of the tax: if buyers can easily do without the good, the price cannot rise much, so sellers pay; if sellers can easily cut production, they will not accept a much lower price, so buyers pay.

In the restaurant market before the tax, the price is \$28 and 94 meals are sold:

$$\varepsilon_S = 3 \times \frac{28}{94} \approx 0.89 \quad \varepsilon_D = -2 \times \frac{28}{94} \approx -0.60 \quad \frac{0.89}{0.89 + 0.60} \approx 0.6$$

The same 60%.

## 4. The Implicit Function Theorem

Section 3 differentiated the equilibrium condition, demand equals supply. Moving supply to the left side gives the same condition in a different form:

$$Q^D(P_b^*(t)) - Q^S(P_b^*(t) - t) = 0$$

We could differentiate this version instead, and get the same answer. Writing an equation with everything on one side and zero on the other is the general form. For the market, the condition is a function of demand and supply:

$$F(Q^D, Q^S) = 0$$

$F$  measures the gap between demand and supply. If there is excess supply (a surplus) or excess demand (a shortage),  $F$  takes a nonzero value.<sup>1</sup> More generally, the same steps work for any equation that links two variables:

$$F(x, y) = 0$$

In the tax example,  $x$  is the tax and  $y$  is the price diners pay. Demand and supply both depend on these two, so  $F$ , demand minus supply, is a function of the tax and the price, and equilibrium requires it to be zero:

$$F(t, P_b) = Q^D(P_b) - Q^S(P_b - t) = 0$$

Suppose that for each value of  $x$ , the equation is satisfied by one value of  $y$ , so  $y$  is a function

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<sup>1</sup>If demand and supply depended only on the price, with no tax or other exogenous variable, the equation would hold at a single price. Nothing outside the market would change, so there would be nothing to compare.

of  $x$  even if we cannot solve for it. How does  $y$  change when  $x$  changes?

When  $x$  changes,  $y$  changes with it so that the equation keeps holding:  $F$  is zero before and after the change. Two things act on  $F$ : the change in  $x$  and the change in  $y$ . The total differential adds the two effects, and since  $F$  does not change, the sum is zero:

$$dF = F_x dx + F_y dy = 0$$

The first term is the effect of the change in  $x$  on  $F$ , the second is the effect of the change in  $y$ , and the two must cancel. Move the first term to the right side:

$$F_y dy = -F_x dx$$

Rearranging the terms gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

In the tax example, the partial derivatives of  $F$  are the slopes of supply and demand. A higher tax lowers quantity supplied, which raises  $F$ , and a higher price lowers demand and raises supply:

$$F_t = \frac{dQ^S}{dP_b} = 3 \quad F_{P_b} = \frac{dQ^D}{dP_b} - \frac{dQ^S}{dP_b} = -2 - 3 = -5$$

The theorem gives the same answer as Section 3:

$$\frac{dP_b^*}{dt} = -\frac{F_t}{F_{P_b}} = -\frac{3}{-5} = 0.6$$

The division needs  $F_y$  to be different from zero. In words,  $y$  must actually affect the equation. If it did not, changing  $y$  could not offset a change in  $x$ .

**Implicit Function Theorem.** Suppose the equation

$$F(x_1, \dots, x_n, y) = 0$$

holds at some point,  $F$  has continuous partial derivatives, and  $F_y$  is not zero there. Then near that point the equation determines  $y$  as a function of  $x_1, \dots, x_n$ , and

$$\frac{\partial y}{\partial x_i} = -\frac{F_{x_i}}{F_y}$$

With several  $x$  variables, we change one at a time and hold the others fixed, so the derivatives become partial derivatives. In this course, the condition on  $F_y$  always holds.

**Example:** A firm produces output with labor  $L$  and machines  $K$ :

$$Q = K^{0.5} L^{0.5}$$

It uses 4 workers and 9 machines, and produces 6 units. If it hires a little more labor, how many machines can it give up per additional worker and still produce 6 units?

The combinations of labor and machines that produce 6 units (the isoquant) satisfy

$$F(L, K) = K^{0.5}L^{0.5} - 6 = 0$$

The partial derivatives are the marginal products:

$$F_L = 0.5 K^{0.5} L^{-0.5} \quad F_K = 0.5 K^{-0.5} L^{0.5}$$

By the theorem:

$$\frac{dK}{dL} = -\frac{F_L}{F_K} = -\frac{0.5 K^{0.5} L^{-0.5}}{0.5 K^{-0.5} L^{0.5}} = -\frac{K}{L} = -\frac{9}{4} = -2.25$$

This is the slope of the isoquant. At 4 workers and 9 machines, for a small increase in labor, each additional worker replaces about 2.25 machines. The number without the minus sign, 2.25, is the marginal rate of technical substitution: the ratio of the marginal products.<sup>2</sup>

Now suppose a worker and a machine cost the same. At the current inputs, a small amount of extra labor replaces 2.25 times as many machines. Spending one more dollar on labor lets the firm stop spending about \$2.25 on machines, and output stays at 6 units. So total cost falls, and the firm can produce 6 units more cheaply by using more labor and fewer machines.

The isoquant is not an equilibrium or optimality condition, and the input mix on it need not be the cheapest. The method works for any equation that links two variables.

## 5. Comparative Statics of an Optimal Choice

More commonly, we differentiate an optimality condition. For a firm or a consumer, the equation to differentiate is the first-order condition for profit or utility maximization. A competitive firm sells output at price  $p$ , has a fixed amount of capital  $\bar{K}$ , and pays wage  $w$ . It hires labor until the value of the last worker's output equals the wage:

$$p \cdot MPL = w$$

Employment is a function of the wage, but the firm does not choose the wage, which is exogenous; it only chooses how many workers to hire. The marginal product depends on employment, and

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<sup>2</sup>The slope is a rate for small changes. For a full extra worker it is only a rough guide. Solving the isoquant directly at 5 workers:

$$\begin{aligned} K^{0.5} \times 5^{0.5} &= 6 \\ K \times 5 &= 36 \quad (\text{square both sides}) \\ K &= 7.2 \end{aligned}$$

The firm gives up 1.8 machines, not 2.25. The isoquant gets flatter as labor rises, so each additional worker replaces fewer machines than the one before, and the slope at 4 workers overstates the trade.

employment depends on the wage:

$$p \cdot MPL(L^*(w)) = w$$

The profit-maximizing condition must hold at every wage, so we differentiate both sides with respect to the wage. The right side has derivative 1. On the left, the marginal product depends on the wage only through employment, so by the chain rule

$$p \cdot \frac{dMPL}{dL} \cdot \frac{dL^*}{dw} = 1 \quad \implies \quad \frac{dL^*}{dw} = \frac{1}{p \cdot \frac{dMPL}{dL}}$$

The marginal product of labor falls as the firm hires more (diminishing marginal returns), so the bottom is negative, and a higher wage lowers employment.

The bottom is also the second derivative of profit, which is negative at a maximum by the second-order condition. This is general with one choice variable: differentiating a first-order condition puts the second derivative of the objective (or minus it, depending on which side of the equation each term is written) in the bottom of the answer. So the second-order condition gives the sign of the bottom. The sign of the answer also depends on the top, which comes from how the exogenous variable enters the first-order condition. Here the wage enters with derivative 1.

**Example:** A bakery has production function

$$Y = K^{0.5} L^{0.5}$$

with 4 ovens. It sells output at \$100 and pays a wage of \$20 per worker. With 4 ovens, output depends only on labor:

$$Y = 4^{0.5} L^{0.5} = 2L^{0.5}$$

The marginal product of labor and its slope are

$$MPL = 2 \times 0.5 L^{-0.5} = L^{-0.5} \quad \frac{dMPL}{dL} = -0.5 L^{-1.5}$$

The first-order condition gives employment:

$$100 \cdot L^{-0.5} = 20$$

$$L^{-0.5} = 0.2$$

$$L^{0.5} = 5$$

$$L^* = 25$$

At 25 workers:

$$25^{1.5} = (25^{0.5})^3 = 5^3 = 125 \quad \frac{dMPL}{dL} = -0.5 \times \frac{1}{125} = -0.004$$

The formula for the wage response gives

$$\frac{dL^*}{dw} = \frac{1}{100 \times (-0.004)} = \frac{1}{-0.4} = -2.5$$

A \$1 increase in the wage cuts employment by about 2.5 workers.

The derivative is a rate of change, so the change in employment is approximately the derivative times the change in the wage. If the wage rises from \$20 to \$22:

$$\Delta L \approx \frac{dL^*}{dw} \times \Delta w = -2.5 \times 2 = -5$$

The prediction is about 25 minus 5, or 20 workers. To check, solve the first-order condition at the new wage:

$$100 \cdot L^{-0.5} = 22$$

$$L^{-0.5} = 0.22$$

$$L^{0.5} = \frac{1}{0.22} \approx 4.545$$

$$L^* \approx 4.545^2 \approx 20.7$$

The prediction is close but not exact. The derivative is the rate at a wage of \$20; as the wage rises, the firm's response gets weaker, so the prediction overstates the drop slightly. The smaller the change in the wage, the closer the prediction.

## Practice Problems

- In the market for cigarettes, at the current equilibrium, the price elasticity of demand is  $-0.4$  and the price elasticity of supply is  $1.2$ .
  - Find the share of a small per-pack tax paid by smokers.
  - The government adds a tax of \$0.50 per pack. By about how much does the price smokers pay rise? How much less do sellers keep per pack?
  - Explain in words why smokers pay most of the tax.
- Demand is  $Q^D = 80 - 3P + 0.4m$  and supply is  $Q^S = 5 + 2P$ , where  $m$  is household income.
  - Differentiate both sides of the equilibrium condition with respect to  $m$  to find  $dP^*/dm$  and  $dQ^*/dm$ .
  - Explain the size of  $dP^*/dm$  using the slopes of demand and supply.
  - At  $m = 50$ , find  $P^*$  and  $Q^*$ . Predict both when  $m$  rises to 60, and check by solving directly.
- Demand is  $Q^D = m/P$  (households spend a fixed amount  $m$  on the good), and supply is  $Q^S = P^2 + 4P$ .

- (a) Show that at  $m = 24$  the equilibrium price is  $P^* = 2$ .
  - (b) Try to solve for  $P^*$  as a function of  $m$ . Why is this hard?
  - (c) Use implicit differentiation to find  $dP^*/dm$  at  $m = 24$ .
  - (d) Predict the equilibrium price at  $m = 25$ , and check that your prediction approximately clears the market.
4. A firm has production function  $Q = K^{1/3}L^{2/3}$  and currently uses  $L = 1$  and  $K = 8$ .
- (a) Use implicit differentiation to find the slope  $dK/dL$  of the isoquant through any point  $(L, K)$ .
  - (b) Evaluate the slope at the firm's current inputs, and explain what the number means.
  - (c) Labor and capital both cost \$1 per unit. Is the firm producing its output at the lowest cost? If not, should it use more labor or more capital?
5. A competitive firm has cost  $C(q) = q^3/3$  and sells at price  $p$ .
- (a) Write the first-order condition for profit maximization.
  - (b) Differentiate it with respect to  $p$  to find  $dq^*/dp$ . Show that the denominator is the second derivative of profit up to a sign, and use the second-order condition to find the sign of  $dq^*/dp$ .
  - (c) At  $p = 16$ , find  $q^*$  and  $dq^*/dp$ . Predict output at  $p = 17$ .
  - (d) Find the price elasticity of the firm's supply at  $p = 16$ , and interpret it.
6. A monopoly faces inverse demand  $P = 100 - 2q$ , has cost  $C(q) = q^2$ , and pays a tax of  $t$  per unit, currently \$10.
- (a) Write the first-order condition, and find output and the price at the current tax.
  - (b) Differentiate the first-order condition with respect to  $t$  to find  $dq^*/dt$ .
  - (c) Use the chain rule to find  $dP^*/dt$ . What share of the tax does the monopoly pass on to its customers?
  - (d) Predict output and the price if the tax rises to \$12.