

14. CONSTRAINED OPTIMIZATION: APPLICATIONS

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EPH 1100: Analytical Tools

Reading: Hoy et al., §13.1 (Example 13.1) and §13.4 (Example 13.6).

- **§13.1, Example 13.1:** The consumer's utility maximization problem (the primal) and expenditure minimization problem (the dual); uncompensated and compensated demand functions; the expenditure function.
- **§13.4, Example 13.6:** Corner solutions in demand analysis.

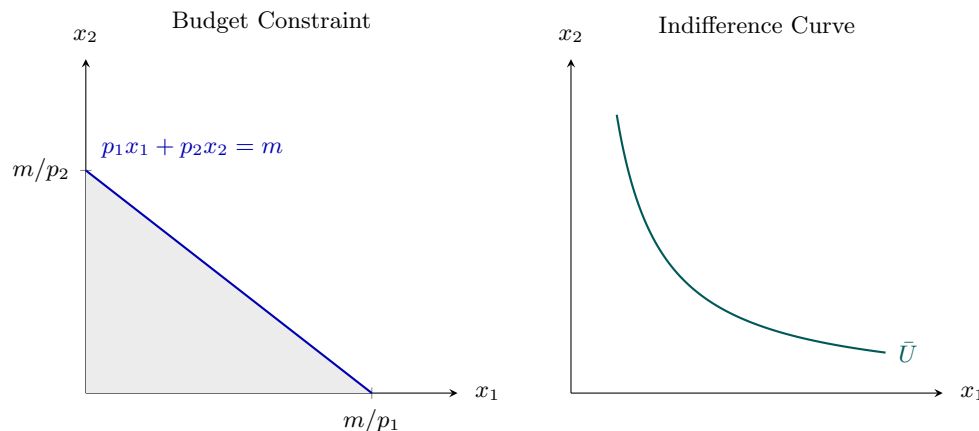
This note applies the Lagrangian method to the two central problems of consumer theory. The first is utility maximization: given an income and market prices, which bundle should the consumer buy? Its solution is the consumer's demand for each good. The second is cost minimization: what is the cheapest way to reach a given level of satisfaction? The two problems are two views of the same choice, and they lead to the same bundle.

1. Utility Maximization

A consumer has income m and faces prices $p_1, \dots, p_n$ for n goods. She can afford any bundle whose total cost does not exceed her income (the budget constraint, left panel of Figure 1), and she ranks bundles by her utility function, whose level curves are the indifference curves (right panel). More of every good is better, so she spends all of her income, and the problem is

$$\max_{x_1, \dots, x_n} U(x_1, \dots, x_n) \quad \text{subject to} \quad \sum_{i=1}^n p_i x_i = m$$

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**Figure 1:** The Consumer's Problem.

Notes: Left: the shaded triangle is every bundle the consumer can afford; the budget line is the set of bundles that cost exactly m , with intercepts m/p_1 and m/p_2 . Right: one indifference curve, the set of bundles that give the same utility $\tilde{U}$.

The Lagrangian is

$$\mathcal{L} = U(x_1, \dots, x_n) + \lambda \left(m - \sum_{i=1}^n p_i x_i \right)$$

With n goods there are $n + 1$ first-order conditions, one per good plus the budget constraint:

$$\begin{aligned} \mathcal{L}_{x_1} &= U_1 - \lambda p_1 = 0 \\ &\vdots \\ \mathcal{L}_{x_n} &= U_n - \lambda p_n = 0 \\ \mathcal{L}_\lambda &= m - \sum_{i=1}^n p_i x_i = 0 \end{aligned}$$

where U_i is the partial derivative of utility with respect to good i , the **marginal utility** of good i (MU_i).

1.1 Tangency and the Multiplier

The two-good results hold for every pair of goods. Dividing the conditions for any two goods i and j gives the tangency condition:

$$\underbrace{\frac{MU_i}{MU_j}}_{\text{MRS}} = \underbrace{\frac{p_i}{p_j}}_{\text{price ratio}}$$

Solving each condition for the multiplier shows that the marginal utility per dollar is the same for every good:

$$\lambda = \frac{MU_1}{p_1} = \frac{MU_2}{p_2} = \dots = \frac{MU_n}{p_n}$$

Their common value is the **marginal utility of income**:

$$\lambda^* \approx \frac{\Delta U^*}{\Delta m}$$

1.2 Marshallian Demands

Solve the tangency conditions for every good in terms of one of them, and substitute into the budget constraint. The result gives each quantity as a function of prices and income, the consumer's **Marshallian** (or uncompensated) **demand** functions:

$$x_i^* = x_i^M(p_1, \dots, p_n, m), \quad i = 1, \dots, n$$

These are the demand curves of introductory economics: they say how much of each good the consumer buys at any prices and income.

Example: Find the Marshallian demands for the Cobb-Douglas utility function

$$U(x_1, x_2) = x_1^\alpha x_2^\beta$$

Step 1: Write the Lagrangian and the first-order conditions.

$$\mathcal{L} = x_1^\alpha x_2^\beta + \lambda(m - p_1 x_1 - p_2 x_2)$$

$$\mathcal{L}_{x_1} = \alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1 = 0$$

$$\mathcal{L}_{x_2} = \beta x_1^\alpha x_2^{\beta-1} - \lambda p_2 = 0$$

$$\mathcal{L}_\lambda = m - p_1 x_1 - p_2 x_2 = 0$$

Step 2: Divide the first condition by the second. The multiplier cancels, and so do most of the powers:

$$\frac{\alpha x_1^{\alpha-1} x_2^\beta}{\beta x_1^\alpha x_2^{\beta-1}} = \frac{p_1}{p_2} \quad \implies \quad \underbrace{\frac{\alpha}{\beta} \cdot \frac{x_2}{x_1}}_{\text{MRS}} = \underbrace{\frac{p_1}{p_2}}_{\text{price ratio}}$$

Rearrange to get spending on good 2 in terms of spending on good 1:

$$p_2 x_2 = \frac{\beta}{\alpha} p_1 x_1$$

Step 3: Substitute into the budget constraint and solve.

$$\begin{aligned} p_1 x_1 + \frac{\beta}{\alpha} p_1 x_1 &= m \\ p_1 x_1 \cdot \frac{\alpha + \beta}{\alpha} &= m \\ x_1^M &= \frac{\alpha}{\alpha + \beta} \cdot \frac{m}{p_1} \end{aligned}$$

By the same steps for good 2:

$$x_2^M = \frac{\beta}{\alpha + \beta} \cdot \frac{m}{p_2}$$

Step 4: Interpret. Multiply each demand by its price to get spending as a share of income:

$$s_1 = \frac{p_1 x_1^M}{m} = \frac{\alpha}{\alpha + \beta}, \quad s_2 = \frac{p_2 x_2^M}{m} = \frac{\beta}{\alpha + \beta}$$

With Cobb-Douglas utility, the consumer spends a fixed share of income on each good, whatever the prices and income. If the exponents are 0.4 and 0.6 and income is \$100, she spends \$40 on good 1 and \$60 on good 2. The shares are fixed, but the quantities are not: if the price of good 1 doubles, she still spends \$40 on it, so she buys half as many units.

2. Elasticities of Demand

Once we have demand functions, we can ask how strongly quantities respond to prices and income. Elasticities measure these responses in percentage terms, so they do not depend on the units in which goods and prices are measured.

Definition. For the Marshallian demand x_i^M :

$$\begin{aligned} \text{Own-price elasticity:} \quad \varepsilon_{ii} &= \frac{\partial x_i^M}{\partial p_i} \cdot \frac{p_i}{x_i} = \frac{\% \Delta \text{ quantity of good } i}{\% \Delta \text{ price of good } i} \\ \text{Cross-price elasticity:} \quad \varepsilon_{ij} &= \frac{\partial x_i^M}{\partial p_j} \cdot \frac{p_j}{x_i} = \frac{\% \Delta \text{ quantity of good } i}{\% \Delta \text{ price of good } j} \\ \text{Income elasticity:} \quad \eta_i &= \frac{\partial x_i^M}{\partial m} \cdot \frac{m}{x_i} = \frac{\% \Delta \text{ quantity of good } i}{\% \Delta \text{ income}} \end{aligned}$$

The signs and sizes have standard names:

- A positive cross-price elasticity means goods i and j are **substitutes** (a higher price of j raises purchases of i); a negative one means they are **complements**.
- A positive income elasticity means good i is a **normal** good; a negative one means it is an **inferior** good.
- A normal good with income elasticity below one is a **necessity** (its budget share falls as income rises); above one, it is a **luxury**.

2.1 Elasticities with Logs

An elasticity is a ratio of percentage changes, and a change in a log is approximately a percentage change. So an elasticity is the derivative of the log of quantity with respect to the log of the price

(or income):

$$\varepsilon_{ij} = \frac{\partial \ln x_i}{\partial \ln p_j}, \quad \eta_i = \frac{\partial \ln x_i}{\partial \ln m}$$

When a demand function is a product of powers, taking logs turns it into a sum, and each elasticity can be read off as a coefficient.

Example: Find the elasticities of the Cobb-Douglas demand for good 1. Write the budget share as s_1 , so that

$$x_1^M = s_1 \cdot \frac{m}{p_1}$$

Step 1: Take logs.

$$\ln x_1^M = \ln s_1 + \ln m - \ln p_1$$

Step 2: Read off the coefficients.

Elasticity	Derivative	Value
Own-price	$\partial \ln x_1 / \partial \ln p_1$	-1
Cross-price	$\partial \ln x_1 / \partial \ln p_2$	0
Income	$\partial \ln x_1 / \partial \ln m$	1

Step 3: Check the own-price elasticity using the definition directly.

$$\varepsilon_{11} = \frac{\partial x_1}{\partial p_1} \cdot \frac{p_1}{x_1} = \left(-s_1 \frac{m}{p_1^2} \right) \cdot \frac{p_1}{s_1 m / p_1} = -s_1 \frac{m}{p_1^2} \cdot \frac{p_1^2}{s_1 m} = -1$$

These values follow from the fixed budget shares. Spending on good 1 does not change when its price changes, so a 1% higher price means a 1% smaller quantity. The price of good 2 does not affect spending on good 1 at all. And spending on good 1 is a fixed share of income, so a 1% higher income means 1% more of the good. The same holds for good 2.

3. Cost Minimization: The Dual Problem

Utility maximization fixes income and asks for the highest utility it can buy. We can turn the question around: fix a level of utility $\bar{U}$, and ask for the cheapest bundle that reaches it.

$$\min_{x_1, \dots, x_n} \sum_{i=1}^n p_i x_i \quad \text{subject to} \quad U(x_1, \dots, x_n) = \bar{U}$$

The original problem is called the **primal**, and this one the **dual**. The two describe the same choice from opposite directions. If \$100 buys a maximum utility of 10, then the cheapest way to reach a utility of 10 costs \$100, and it is the same bundle.

The dual is useful because it holds utility fixed. When a price rises, it answers the question “how much more income would the consumer need to be as well off as before?”, which is the question behind cost-of-living adjustments.

The Lagrangian (using μ for the multiplier, to keep it apart from the primal's λ) is

$$\mathcal{L} = \sum_{i=1}^n p_i x_i + \mu(\bar{U} - U(x_1, \dots, x_n))$$

with first-order conditions

$$\begin{aligned}\mathcal{L}_{x_1} &= p_1 - \mu U_1 = 0 \\ &\vdots \\ \mathcal{L}_{x_n} &= p_n - \mu U_n = 0 \\ \mathcal{L}_\mu &= \bar{U} - U(x_1, \dots, x_n) = 0\end{aligned}$$

Dividing the conditions for any two goods gives the same tangency condition as the primal:

$$\frac{MU_i}{MU_j} = \frac{p_i}{p_j}$$

Only the last condition is different: the bundle must now reach the target utility, instead of exhausting the budget.

3.1 The Multiplier

Solve any condition for μ :

$$\mu = \frac{p_1}{MU_1} = \frac{p_2}{MU_2} = \dots = \frac{p_n}{MU_n}$$

Each ratio is the cost of one more unit of utility obtained through that good. So μ^* is the **marginal cost of utility**: the extra spending needed to raise the target utility by one unit. It is the reciprocal of the primal's multiplier, which measured utility per dollar:

$$\lambda^* \approx \frac{\Delta U}{\Delta m}, \quad \mu^* \approx \frac{\Delta m}{\Delta U}, \quad \lambda^* = \frac{1}{\mu^*}$$

3.2 Hicksian Demands

In the primal we substituted the tangency condition into the budget constraint. In the dual we substitute it into the utility constraint. The result gives each quantity as a function of prices and the target utility, the **Hicksian** (or compensated) **demand** functions:

$$x_i^* = x_i^H(p_1, \dots, p_n, \bar{U}), \quad i = 1, \dots, n$$

Adding up the cost of the Hicksian bundle gives the **expenditure function**, the least it costs to reach $\bar{U}$ at the given prices:

$$E(p_1, \dots, p_n, \bar{U}) = \sum_{i=1}^n p_i x_i^H(p_1, \dots, p_n, \bar{U})$$

Example: A consumer has utility

$$U(x_1, x_2) = x_1^{0.5} x_2^{0.5}$$

and faces prices of 2 for good 1 and 4 for good 2. In the primal, with income of 100, she buys 25 units of good 1 and 12.5 units of good 2, reaches a utility of 17.68, and has a marginal utility of income of 0.177. We find her Hicksian demands and show that the dual gives back the same bundle.

Step 1: Write the Lagrangian and the first-order conditions, with prices left general.

$$\mathcal{L} = p_1 x_1 + p_2 x_2 + \mu (\bar{U} - x_1^{0.5} x_2^{0.5})$$

$$\mathcal{L}_{x_1} = p_1 - 0.5 \mu x_1^{-0.5} x_2^{0.5} = 0$$

$$\mathcal{L}_{x_2} = p_2 - 0.5 \mu x_1^{0.5} x_2^{-0.5} = 0$$

$$\mathcal{L}_\mu = \bar{U} - x_1^{0.5} x_2^{0.5} = 0$$

Step 2: Divide the first condition by the second. The multiplier cancels:

$$\frac{x_2}{x_1} = \frac{p_1}{p_2} \quad \implies \quad x_2 = \frac{p_1}{p_2} x_1$$

Step 3: Substitute into the utility constraint (not the budget constraint) and solve.

$$\begin{aligned} x_1^{0.5} \left(\frac{p_1}{p_2} x_1 \right)^{0.5} &= \bar{U} \\ x_1 \left(\frac{p_1}{p_2} \right)^{0.5} &= \bar{U} \\ x_1^H &= \bar{U} \left(\frac{p_2}{p_1} \right)^{0.5} \end{aligned}$$

By the same steps for good 2:

$$x_2^H = \bar{U} \left(\frac{p_1}{p_2} \right)^{0.5}$$

Income does not appear: these demands hold utility fixed, and spending adjusts to whatever reaching $\bar{U}$ costs.

Step 4: Find the expenditure function.

$$E = p_1 \bar{U} \left(\frac{p_2}{p_1} \right)^{0.5} + p_2 \bar{U} \left(\frac{p_1}{p_2} \right)^{0.5} = \bar{U} (p_1 p_2)^{0.5} + \bar{U} (p_1 p_2)^{0.5} = 2 \bar{U} (p_1 p_2)^{0.5}$$

Step 5: Plug in the numbers. With prices of 2 and 4, and target utility equal to the primal's maximum utility,

$$\bar{U} = 25^{0.5} \cdot 12.5^{0.5} = \sqrt{312.5} \approx 17.68$$

the Hicksian demands and the expenditure function give

$$\begin{aligned}x_1^H &= 17.68 \cdot \left(\frac{4}{2}\right)^{0.5} = 17.68 \cdot 1.414 \approx 25 \\x_2^H &= 17.68 \cdot \left(\frac{2}{4}\right)^{0.5} = 17.68 \cdot 0.707 \approx 12.5 \\E &= 2 \cdot 17.68 \cdot (2 \cdot 4)^{0.5} = 2 \cdot 17.68 \cdot 2.828 \approx 100\end{aligned}$$

The cheapest way to reach the utility that \$100 could buy costs exactly \$100, and it is the same bundle.

Step 6: Find the multiplier. From the first condition, evaluated at the optimal bundle:

$$\mu^* = \frac{p_1}{0.5 x_1^{-0.5} x_2^{0.5}} = \frac{2}{0.5 \cdot (12.5/25)^{0.5}} = \frac{2}{0.354} \approx 5.66$$

Raising utility by one unit costs about \$5.66. Its reciprocal is the primal's marginal utility of income:

$$\frac{1}{\mu^*} = \frac{1}{5.66} \approx 0.177 = \lambda^*$$

It also matches the expenditure function: with prices of 2 and 4, spending rises by $2(8)^{0.5}$ for each unit of utility, which is 5.66.

3.3 The Firm's Version

A firm that must produce $\bar{Q}$ units at the lowest cost solves a problem with the same structure, with inputs in place of goods and a production function in place of the utility function:

$$\min_{K,L} wL + rK \quad \text{subject to} \quad F(K, L) = \bar{Q}$$

$$\mathcal{L} = wL + rK + \mu(\bar{Q} - F(K, L))$$

Each part of the consumer's dual has a counterpart for the firm. Substituting the tangency condition into the production constraint gives the **conditional input demands**, the firm's counterpart of the Hicksian demands.

	Consumer (dual)	Firm
Objective	Minimize spending	Minimize cost
Constraint	Reach utility $\bar{U}$	Produce output $\bar{Q}$
Tangency	MRS equals price ratio	MRTS equals input price ratio
Multiplier	Marginal cost of utility	Marginal cost of output
Solution	Hicksian demands	Conditional input demands

Practice Problems

1. **Cobb-Douglas.** Use the Lagrangian to find the Marshallian demands for each of the following utility functions.

- (a) $U = 3x_1^{0.25}x_2^{0.75}$. If $p_1 = 5$, $p_2 = 10$, and $m = 500$, how much does the consumer buy of each good? What is her utility?
- (b) $U = \ln 3 + 0.25 \ln x_1 + 0.75 \ln x_2$, the log of the utility function in (a). Taking the log is a monotonic transformation.
- Find the Marshallian demands at the prices and income in (a).
 - Are they the same as in (a)? Do the two utility functions give the same utility at that bundle? Does it matter?
- (c) For the Cobb-Douglas demands in Section 1.2, find the own-price, cross-price, and income elasticities of good 2.

2. **Perfect Complements.** A consumer has utility

$$U(x_1, x_2) = \min\{3x_1, 2x_2\}$$

- (a) With $p_1 = 1$, $p_2 = 2$, and $m = 80$, how much does the consumer buy of each good? What is her utility? (*Hint:* do not use the Lagrangian.)
- (b) Find the Hicksian demands and the expenditure function for general prices p_1, p_2 and target utility $\bar{U}$.
- (c) Plug the prices from (a) and the utility she reached in (a) into the expenditure function. Does it give back her income?

3. **Perfect Substitutes.** A consumer has utility

$$U(x_1, x_2) = x_1 + 5x_2$$

and faces prices $p_1 = 1$ and $p_2 = 2$, and income $m = 12$.

- (a) How much does the consumer buy of each good? (*Hint:* do not use the Lagrangian.)
- (b) The price of good 2 rises while the price of good 1 and income stay the same. At what price of good 2 does the consumer switch to good 1?
- (c) Sketch her demand for good 2 as a function of its price. What happens to the quantity at the price you found in (b)?

4. **Stone-Geary Utility.** A consumer has utility

$$U(x_1, x_2) = (x_1 - \gamma_1)^\alpha (x_2 - \gamma_2)^\beta, \quad \alpha + \beta = 1, \quad x_i > \gamma_i$$

This is a Cobb-Douglas function shifted by subsistence quantities γ_1 and γ_2 : the consumer must first secure γ_i units of each good, and gets Cobb-Douglas utility from whatever she consumes beyond that.

- (a) Let $\alpha = \beta = 0.5$, $\gamma_1 = 2$, and $\gamma_2 = 1$. Find the Marshallian demands.
- (b) Find the Marshallian demands for general α , β , γ_1 , and γ_2 . Show that you recover the Cobb-Douglas demands from Section 1.2 when $\gamma_1 = \gamma_2 = 0$.
- (c) Interpret: how is income split between the two goods?

5. **Constant Elasticity of Substitution (CES) Utility.** A consumer has utility

$$U(x_1, x_2) = (\alpha x_1^\rho + \beta x_2^\rho)^{1/\rho}, \quad \rho < 1, \quad \rho \neq 0$$

The parameter ρ governs how easily the consumer substitutes one good for the other, and α and β govern how much she values each good. When ρ is positive the goods are closer to substitutes, and when it is negative they are closer to complements.

- (a) Find the Marshallian demands when ρ , α , and β are all equal to 0.5.
- (b) Find the Marshallian demands for general ρ , α , and β .
- (c) Unlike the Cobb-Douglas case, does the demand for good 1 depend on the price of good 2?

6. **Primal and Dual.** Pick a Cobb-Douglas utility function

$$U(x_1, x_2) = A x_1^\alpha x_2^\beta$$

where the exponents add up to one but are not both 0.5, and A is any positive number (say 5 or 10).

- (a) **Utility maximization.** Set up the Lagrangian for maximizing utility subject to the budget constraint $p_1 x_1 + p_2 x_2 = m$, and find the Marshallian demands.
- (b) **Cost minimization.** Set up the Lagrangian for minimizing $p_1 x_1 + p_2 x_2$ subject to reaching utility $\bar{U}$, and find the Hicksian demands.
- (c) **Same bundle.** Pick prices (say $p_1 = 5$ and $p_2 = 10$) and an income m . Use the Marshallian demands to find the quantities, and plug them into the utility function to find $\bar{U}$. Show that the Hicksian demands at the same prices and this $\bar{U}$ give the same quantities.
- (d) **Expenditure function.** Substitute the Hicksian demands into $p_1 x_1 + p_2 x_2$ to find the expenditure function $E(p_1, p_2, \bar{U})$, and simplify it as much as you can.
- (e) **Derivatives of E .** Take the partial derivatives of E with respect to p_1 and p_2 . Compare them with the Hicksian demands from (b).

- (f) **The role of A .** Pick a different value of A and repeat (a) and (b). Show that changing A changes the utility reached with a given income, and the spending needed to reach a given $\bar{U}$, but does not change the Marshallian demands. Explain why.

7. **Firm Cost Minimization.** A firm has production function

$$Q = K^{1/3}L^{2/3}$$

It pays a wage $w = 6$ and a rental rate $r = 3$, and must produce $\bar{Q} = 12$ units.

- (a) Set up the cost-minimization Lagrangian and write the first-order conditions.
- (b) Find the cost-minimizing ratio of capital to labor.
- (c) Solve for L^* and K^* , and compute the minimum cost C^* .
- (d) Compute μ^* and interpret it.
- (e) Re-solve the problem with $\bar{Q} = 13$. By how much does the minimum cost rise? Compare with your answer to (d).