

11. UNCONSTRAINED OPTIMIZATION: FIRST-ORDER CONDITIONS

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EPH 1100: Analytical Tools

Reading: Hoy et al., §12.1.

- **§12.1:** Necessary conditions for unconstrained interior optima; first-order conditions for functions of two variables; solving the stationary-point system.

1. Introduction

When a firm controls a single choice variable, we found the profit-maximizing output by setting the derivative to zero and checking the second-order condition. That logic still applies with more than one choice variable: set every partial derivative to zero, then check second-order conditions. There are more equations to write and more conditions to check, but the steps themselves do not change.

Real economic decisions rarely involve just one choice variable. A firm choosing how much capital K and labor L to employ cannot optimize one in isolation: hiring more labor changes how productive an additional unit of capital is, and vice versa, so the profit-maximizing bundle of the two has to be found jointly, not one input at a time.

This note covers the first-order conditions: how to find candidate optimal points. Classifying each candidate as a maximum, minimum, or saddle point comes later.

2. Stationary Points for Functions of Two Variables

Recall the single-variable case: x^* is a stationary point of $f(x)$ if $f'(x^*) = 0$. At a local maximum or minimum, the function is momentarily flat in the only direction available.

For a function $f(x, y)$, a local maximum or minimum must be flat in every direction simultaneously. In particular, it must be flat when you move only in the x -direction (holding y fixed), and flat when you move only in the y -direction (holding x fixed). Those conditions are exactly the partial derivatives.

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Definition. A point (x^*, y^*) is a **stationary point** of $f(x, y)$ if both partial derivatives are zero simultaneously:

$$f_x(x^*, y^*) = 0 \quad \text{and} \quad f_y(x^*, y^*) = 0$$

Equivalently, $\nabla f(x^*, y^*) = \mathbf{0}$. This is also commonly called a **critical point**; the two terms are interchangeable.

Every local maximum and every local minimum of a smooth function is a stationary point. But not every stationary point is a local maximum or minimum. There is a third possibility, the saddle point: flat in every direction at that point, yet curving upward in some directions and downward in others, the way a mountain pass rises toward the peaks on either side but falls away toward the valleys in front and behind. Figure 1 shows one. We will examine saddle points closely later; for now, the task is finding stationary points.

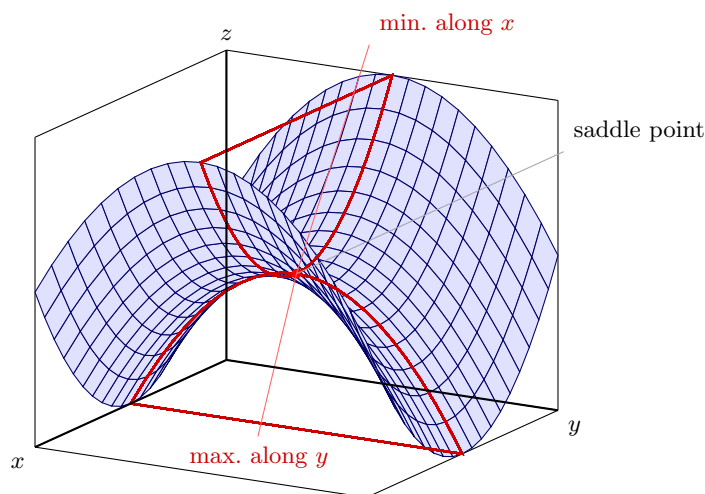


Figure 1: A Saddle Point.

Notes: The surface $z = x^2 - y^2$ is flat at the origin (both partial derivatives are zero there). The red curve at $y = 0$ shows that z has a minimum along the x -direction; the red curve at $x = 0$ shows that z has a maximum along the y -direction. A point that is a minimum in one direction and a maximum in another is a saddle point.

3. The First-Order Conditions

To find the stationary points of $f(x, y)$, we set both partial derivatives to zero and solve the resulting two-equation system. These are the **first-order conditions (FOC)**.

The general procedure:

Step 1: Compute f_x and f_y .

Step 2: Write the system $f_x = 0$ and $f_y = 0$.

Step 3: Solve the system for (x^*, y^*) .

Step 4: Evaluate f at the stationary point(s).

When the objective function is quadratic, the partial derivatives are linear in x and y , and the system in Step 3 is a 2×2 linear system. Substitution is typically the fastest route.

Example. A firm sells into two separate markets, domestic (x) and export (y), which it can price independently. Both markets are supplied from the same production line, so cost depends on total output $Q = x + y$. Profit is total revenue from both markets minus this shared cost:

$$\pi(x, y) = \underbrace{xP_A(x)}_{R_A(x)} + \underbrace{yP_B(y)}_{R_B(y)} - C(x + y)$$

Suppose inverse demand is $P_A(x) = 14 - x$ in the domestic market and $P_B(y) = 10 - y$ in the export market, and the shared cost function is $C(Q) = 0.5Q^2$. Substituting:

$$\pi(x, y) = x(14 - x) + y(10 - y) - 0.5(x + y)^2 = 14x + 10y - 1.5x^2 - 1.5y^2 - xy$$

Step 1: Compute the partial derivatives.

$$\pi_x = 14 - 3x - y \quad \pi_y = 10 - x - 3y$$

Step 2: Set both equal to zero.

$$3x + y = 14 \quad \text{and} \quad x + 3y = 10$$

Step 3: Solve the system. From the first equation: $y = 14 - 3x$. Substitute into the second:

$$x + 3(14 - 3x) = 10 \implies x + 42 - 9x = 10 \implies -8x = -32 \implies x^* = 4$$

Then $y^* = 14 - 3(4) = 2$.

Step 4: Evaluate profit at the candidate optimum.

$$\pi(4, 2) = 14(4) + 10(2) - 1.5(4^2) - 1.5(2^2) - (4)(2) = 56 + 20 - 24 - 6 - 8 = 38$$

The only stationary point is $(x^*, y^*) = (4, 2)$, with $\pi^* = 38$.

At this point, we have found the *candidate* for the profit maximum. Whether it is a true maximum (rather than a minimum or saddle point) requires second-order analysis, which comes later.

3.1 Interpreting the FOC

Each first-order condition has a straightforward economic interpretation. Recall $\pi(x, y) = R_A(x) + R_B(y) - C(x + y)$. Differentiating termwise:

$$\pi_x = R'_A(x) - C'(x + y) \quad \pi_y = R'_B(y) - C'(x + y)$$

The term $C'(x + y)$ shows up in *both* conditions, because the marginal cost depends on the level of joint production of x and y . Put differently, this is just the marginal cost of production, $C'(Q)$, where $Q = x + y$. From the point of view of production costs, it doesn't matter whether it will be sold in the domestic or export market. This is exactly the channel through which the two choice variables interact.

Setting each condition to zero gives $R'_A(x) = C'(Q)$ and $R'_B(y) = C'(Q)$: marginal revenue in each market equals the shared marginal cost of production, the familiar MR = MC rule, applied separately to each market but tied together through a common cost of output.

For the functions in this example, $R'_A(x) = 14 - 2x$, $R'_B(y) = 10 - 2y$, and $C'(Q) = Q = x + y$, so the two FOCs read $14 - 2x = x + y$ and $10 - 2y = x + y$ — the same equations found in Step 2, rearranged to put marginal revenue on the left and marginal cost on the right.

3.2 Extension to n Variables

For a function of n variables $f(x_1, \dots, x_n)$, the FOC is a system of n equations:

$$\frac{\partial f}{\partial x_i} = 0 \quad \text{for } i = 1, 2, \dots, n$$

The intuition is unchanged: at a local maximum or minimum, the function must be flat in every coordinate direction. More choice variables means a larger system to solve, but the approach is identical.

Practice Problems

1. Consider $f(x, y) = x^2 + y^2 - 4x - 6y + 5$.
 - (a) Find the stationary point of f .
 - (b) Does the FOC tell you whether this is a maximum or minimum? What additional information would you need?
2. A firm's profit depends on output q and advertising spending a :

$$\pi(q, a) = 20q - q^2 + 16a - a^2 - qa$$

- (a) Find π_q and π_a .

- (b) Set up and solve the FOC system for (q^*, a^*) .
 - (c) Evaluate $\pi(q^*, a^*)$.
 - (d) Interpret each first-order condition economically.
3. A government sets tax rates on two goods, t_1 and t_2 , to maximize total tax revenue:

$$R(t_1, t_2) = 8t_1 + 9t_2 - 2t_1^2 - t_2^2 - t_1t_2$$

- (a) Find R_{t_1} and R_{t_2} .
- (b) Find the stationary point (t_1^*, t_2^*) and compute $R(t_1^*, t_2^*)$.