

9. LINEAR ALGEBRA: SOLVING SYSTEMS AND DETERMINANTS

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EPH 1100: Analytical Tools

Reading: Hoy et al., §7.1 and §9.1.

- **§7.1:** Systems of two linear equations; systems with no solution, one solution, or infinitely many solutions.
- **§9.1 (2×2 case only):** The inverse and the determinant of a 2×2 matrix.

1. A Market as a System of Equations

A market has a demand curve and a supply curve:

$$Q^d = a - bP, \quad Q^s = -c + dP$$

The parameters a , b , c , and d are positive numbers. Demand slopes down and supply slopes up. In equilibrium, the quantity demanded equals the quantity supplied. Call that common quantity Q . The equilibrium is then a pair of numbers, P and Q , that satisfies both equations at once:

$$Q = a - bP \quad (\text{demand})$$

$$Q = -c + dP \quad (\text{supply})$$

This is a system of two equations in two unknowns. You already know how to solve it by substitution or elimination. Setting the two right-hand sides equal gives:

$$P^* = \frac{a + c}{b + d}, \quad Q^* = \frac{ad - bc}{b + d}$$

The rest of this note solves the same system with linear algebra.

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2. Solving with Linear Algebra

Step 1: Write the system in matrix form. Put the unknowns on the left and the constants on the right:

$$\begin{aligned} bP + Q &= a \\ -dP + Q &= -c \end{aligned}$$

This is one matrix equation:

$$\underbrace{\begin{pmatrix} b & 1 \\ -d & 1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} P \\ Q \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} a \\ -c \end{pmatrix}}_{\mathbf{k}}$$

A is the **coefficient matrix**, $\mathbf{x}$ is the **vector of unknowns**, and $\mathbf{k}$ is the **vector of constants**. In short:

$$A\mathbf{x} = \mathbf{k}$$

Step 2: Divide by A . With one equation in one unknown, we would divide both sides by the coefficient:

$$ax = k \quad \implies \quad x = \frac{k}{a}$$

Matrices have no division. Instead, we multiply by the **inverse**.

Definition. The **inverse** of a square matrix A , written A^{-1} , is the matrix that undoes multiplication by A :

$$A^{-1}A = I$$

where I is the identity matrix.

Multiplying both sides of the system by A^{-1} solves it:

$$\begin{aligned} A\mathbf{x} &= \mathbf{k} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{k} \\ I\mathbf{x} &= A^{-1}\mathbf{k} \\ \mathbf{x} &= A^{-1}\mathbf{k} \end{aligned}$$

Step 3: Find the inverse. For a 2×2 matrix, the inverse has a short formula. It uses a number called the determinant.

The 2×2 determinant and inverse. For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the **determinant** is

$$\det A = ad - bc$$

and, if the determinant is not zero, the inverse is

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

In words: swap the two diagonal entries, change the sign of the two off-diagonal entries, and divide every entry by the determinant.

For the market:

$$\det A = (b)(1) - (1)(-d) = b + d$$

$$A^{-1} = \frac{1}{b+d} \begin{pmatrix} 1 & -1 \\ d & b \end{pmatrix}$$

Step 4: Multiply.

$$\underbrace{\begin{pmatrix} P^* \\ Q^* \end{pmatrix}}_{\mathbf{x}} = \underbrace{\frac{1}{b+d} \begin{pmatrix} 1 & -1 \\ d & b \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} a \\ -c \end{pmatrix}}_{\mathbf{k}} = \frac{1}{b+d} \begin{pmatrix} a+c \\ ad-bc \end{pmatrix}$$

This is the same answer as in Section 1:

$$P^* = \frac{a+c}{b+d}, \quad Q^* = \frac{ad-bc}{b+d}$$

With numbers. Suppose demand and supply are:

$$Q^d = 100 - 2P, \quad Q^s = -20 + 3P$$

Then:

$$A = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}, \quad \det A = 2 + 3 = 5, \quad A^{-1} = \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} P^* \\ Q^* \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 100 \\ -20 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 100 + 20 \\ 300 - 40 \end{pmatrix} = \begin{pmatrix} 24 \\ 52 \end{pmatrix}$$

Check with the original equations: demand gives 100 minus 48, or 52, and supply gives 72 minus 20, or 52.

3. When There Is No Unique Solution

The inverse formula divides by the determinant. If the determinant is zero, the inverse does not exist, and the system has no unique solution.

Graphically, each equation is a line, and a solution is a point on both lines. Two lines can cross once, be parallel and never meet, or be the same line (Figure 1). A zero determinant means the two lines have the same slope, so they are parallel or identical.

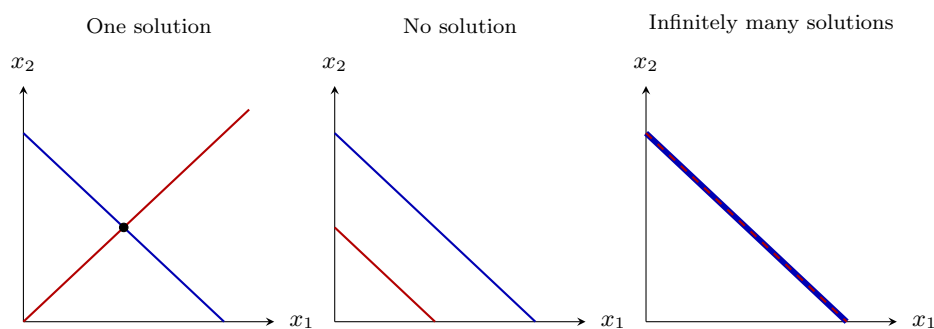


Figure 1: The Three Possible Outcomes for Two Linear Equations.

Notes: Left: $x_1 + x_2 = 4$ and $x_1 - x_2 = 0$ cross once, at $(2, 2)$. Middle: $x_1 + x_2 = 4$ and $x_1 + x_2 = 2$ have the same slope and never meet. Right: $x_1 + x_2 = 4$ and $2x_1 + 2x_2 = 8$ describe the same line, so every point on it solves both.

In the market, the determinant is the sum of the two slopes, b and d . Both are positive, so the determinant is never zero: a downward-sloping demand curve and an upward-sloping supply curve always cross exactly once.

The 2×2 determinant also matters for optimization with two variables: it is the test for whether a stationary point is a maximum, a minimum, or neither.

4. Many Equations and Many Unknowns

For one market, substitution is as fast as the inverse. The payoff from linear algebra comes with larger systems. With n equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= k_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= k_n \end{aligned}$$

Substitution and elimination now take many rounds, each rewriting several equations, and one arithmetic mistake at any step makes the answer wrong. In matrix form, the system is the same as before:

$$\underbrace{A}_{n \times n} \underbrace{\mathbf{x}}_{n \times 1} = \underbrace{\mathbf{k}}_{n \times 1}$$

and so is the solution:

$$\mathbf{x} = A^{-1}\mathbf{k}$$

Steps 1 to 4 of Section 2 do not change: write A and $\mathbf{k}$, check that the determinant of A is not zero, find A^{-1} , and multiply. The only difference is that a large inverse is computed by software, not by hand. In Python, the inverse is `np.linalg.inv(A)`, which the regression homework uses.

The most important example in economics is linear regression, where the system has hundreds or thousands of equations.

5. Application: Linear Regression

Linear regression, the main tool of econometrics, is a system of linear equations written in matrix form.

Example: A labor economist models the hourly wage of worker i as a function of years of schooling and years of experience:

$$\text{wage}_i = b_0 + b_1 \text{school}_i + b_2 \text{exper}_i + e_i$$

The coefficients b_0 , b_1 , and b_2 are the same for every worker. The error e_i is the part of worker i 's wage that schooling and experience do not explain.

Step 1: Write one equation per worker. For the first three workers in the data:

$$22.50 = b_0 + 12 b_1 + 5 b_2 + e_1$$

$$31.00 = b_0 + 16 b_1 + 8 b_2 + e_2$$

$$18.75 = b_0 + 12 b_1 + 2 b_2 + e_3$$

Step 2: Stack the equations. The wages go in a column vector $\mathbf{y}$, the coefficients in $\mathbf{b}$, and the errors in $\mathbf{e}$. The data go in the matrix X , with one row per worker and one column per coefficient. The first column is all ones, because b_0 enters every equation with a coefficient of 1:

$$\underbrace{\begin{pmatrix} 22.50 \\ 31.00 \\ 18.75 \\ \vdots \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} 1 & 12 & 5 \\ 1 & 16 & 8 \\ 1 & 12 & 2 \\ \vdots & \vdots & \vdots \end{pmatrix}}_X \underbrace{\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}}_{\mathbf{b}} + \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{pmatrix}}_{\mathbf{e}}$$

With 500 workers, X is 500×3 , and the whole model is

$$\mathbf{y} = X\mathbf{b} + \mathbf{e}$$

Here the data in X and $\mathbf{y}$ are known, and the unknowns are the coefficients in $\mathbf{b}$.

Transpose

Solving for $\mathbf{b}$ needs one more definition.

Definition. The **transpose** of a matrix A , written A' (“ A prime”), turns the rows of A into columns. If A is $m \times n$, then A' is $n \times m$:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad A' = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

The Regression Formula

With single numbers, we would solve for b by dividing y by x :

$$y = xb \quad \implies \quad b = \frac{y}{x}$$

With matrices, dividing means multiplying by an inverse (Section 2). But X is 500×3 , not square, so it has no inverse. The fix is to multiply both sides by X' first. The method of **least squares**, which chooses $\mathbf{b}$ to make the errors as small as possible, leads to exactly this system:

$$X'X \mathbf{b} = X'\mathbf{y}$$

The dimensions show why the multiplication has to be done in this order:

$$\underbrace{X'}_{3 \times 500} \underbrace{X}_{500 \times 3} = \underbrace{X'X}_{3 \times 3}, \quad \underbrace{X'}_{3 \times 500} \underbrace{\mathbf{y}}_{500 \times 1} = \underbrace{X'\mathbf{y}}_{3 \times 1}$$

$X'X$ is square, with one row and one column per coefficient, so it can have an inverse. Multiplying both sides by that inverse gives the regression formula:

$$\mathbf{b} = (X'X)^{-1}X'\mathbf{y}$$

Multiplying in the other order would not work: XX' is 500×500 , and $\mathbf{y}X'$ is not defined.

Perfect collinearity. The formula requires $X'X$ to have an inverse, so its determinant cannot be zero. That fails when one column of X is an exact combination of others. For example, a regression with a constant and a dummy for each of the four quarters has a column of ones equal to the sum

of the four dummy columns (the **dummy variable trap**). With one observation per quarter:

$$X = \begin{matrix} & \text{const} & Q_1 & Q_2 & Q_3 & Q_4 \\ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{matrix}$$

In every row, the four quarter dummies add up to 1, the value of the constant. Statistical software detects this and drops one of the dummies.

Practice Problems

1. A market has demand $Q^d = 50 - 2P$ and supply $Q^s = -10 + 4P$.
 - (a) Write the equilibrium conditions as a system in the form $A\mathbf{x} = \mathbf{k}$, with $\mathbf{x} = (P, Q)$.
 - (b) Compute $\det A$ and A^{-1} .
 - (c) Use A^{-1} to find the equilibrium price and quantity.
 - (d) Check your answer by substitution.
2. Return to the general market in Section 1, with demand $Q^d = a - bP$ and supply $Q^s = -c + dP$.
 - (a) Suppose demand shifts out, so that a rises by 1. Which of A and $\mathbf{k}$ changes?
 - (b) Use A^{-1} to find how much P^* and Q^* rise.
 - (c) Explain why you did not need to compute a new inverse.
3. A student sets up a model and obtains the equations

$$2P_1 - P_2 = 10, \quad 4P_1 - 2P_2 = 20$$

- (a) Compute the determinant of the coefficient matrix.
- (b) Show that one equation is a multiple of the other. How many solutions does the system have?
- (c) Suppose the right-hand side of the second equation were 30 instead. How many solutions would the system have?