

8. LINEAR ALGEBRA: VECTORS AND MATRICES

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EPH 1100: Analytical Tools

Reading: Hoy et al., §8.1–8.2 and §7.3.

- **§8.1:** Matrix notation and dimension; square, diagonal, identity, and zero matrices.
- **§8.2:** Addition, subtraction, scalar multiplication, and matrix multiplication.
- **§7.3 (up to the length of a vector):** Row and column vectors; the inner (dot) product.

1. Why Linear Algebra?

Economic data often has more than one dimension. For instance, a trade analyst tracking bilateral export flows needs to record who exports how much to whom, so each value is tied to two countries at once. Similarly, a human resources department records wages by job level and by department. In such cases, the data fills a rectangular grid, with one dimension along the rows and another along the columns.

Matrix algebra is the branch of mathematics that gives systematic rules for working with numbers arranged in rows and columns. Rather than performing each cell's calculation separately, matrix operations let us add, scale, and multiply entire tables in compact expressions that work the same way for tables of any size.

Example: Three countries trade with each other. The table below records annual export values in billions of dollars. Each row is an exporting country and each column is an importing country:

Exporter	Importer		
	Country 1	Country 2	Country 3
Country 1	0	12	8
Country 2	5	0	15
Country 3	10	7	0

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Dropping the labels and keeping the numbers gives the matrix T :

$$T = \begin{pmatrix} 0 & 12 & 8 \\ 5 & 0 & 15 \\ 10 & 7 & 0 \end{pmatrix}$$

The entry in row i and column j , written t_{ij} , is the value of exports from country i to country j . For instance, t_{12} sits in row 1, column 2: country 1 exports \$12 billion to country 2. In the opposite direction, t_{21} sits in row 2, column 1: country 2 exports only \$5 billion to country 1.

Diagonal entries are zero (no country exports to itself). Row sums give each country's total exports; column sums give total imports. Suppose all export prices fall by 10%. Then every entry scales by 0.9: a single scalar multiplication $0.9 \cdot T$, rather than nine separate adjustments. If we want to compare this year's flows to last year's, we add the two matrices entry by entry.

2. Matrices

Definition. A **matrix** is a rectangular array of numbers, enclosed in brackets or parentheses. A matrix with m rows and n columns has **dimension** $m \times n$. The entry in row i , column j is written a_{ij} .

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

A **column vector** is an $n \times 1$ matrix (one column, many rows). A **row vector** is a $1 \times n$ matrix. We will write vectors in bold: $\mathbf{x}$, $\mathbf{b}$, $\mathbf{p}$.

Special Matrices

- **Square matrix:** same number of rows and columns ($n \times n$).
- **Identity matrix** I_n : an $n \times n$ square matrix with 1s on the main diagonal and 0s everywhere else. It plays the role of the number 1 in matrix algebra: multiplying an $n \times n$ matrix A by I_n leaves it unchanged.

$$AI_n = I_nA = A$$

For two rows and two columns:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- **Zero matrix** 0 : a matrix of any dimension whose entries are all zero. It plays the role of the number 0.

- **Diagonal matrix:** a square matrix whose entries off the main diagonal are all zero. For example:

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

The identity matrix is the diagonal matrix with every diagonal entry equal to 1.

- **Symmetric matrix:** a square matrix whose entries mirror each other across the main diagonal, so the entry in row i , column j equals the entry in row j , column i :

$$a_{ij} = a_{ji}$$

For example:

$$S = \begin{pmatrix} 2 & 5 & -1 \\ 5 & 0 & 3 \\ -1 & 3 & 4 \end{pmatrix}$$

3. Matrix Operations

Scalar Operations

A **scalar** is a single real number. Multiplying a matrix by a scalar multiplies every entry of the matrix by that scalar:

$$3 \begin{pmatrix} 1 & 4 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 12 \\ 6 & 0 \end{pmatrix}$$

Dividing by a scalar is the same operation, since dividing by 2 is multiplying by one half.

Adding a scalar to a matrix is not a standard operation. To add the same number to every entry, we add a matrix of the same dimension with that number in every cell, using the addition rule below.

Addition and Subtraction

Two matrices can be added or subtracted only if they have the same dimension. The operation is performed element by element.

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 1 & 10 \end{pmatrix}$$

The Dot Product

Matrix multiplication is built from a simpler operation between two vectors.

Definition. The **dot product** (or **inner product**) of two vectors with the same number of entries multiplies them entry by entry and adds the results:

$$\mathbf{p} \cdot \mathbf{x} = p_1x_1 + p_2x_2 + \cdots + p_nx_n$$

The result is a single number.

Example: A consumer buys 5 loaves of bread, 3 cartons of milk, and 2 bags of coffee, at prices of \$4, \$3, and \$6. With the quantities in $\mathbf{x}$ and the prices in $\mathbf{p}$, total spending is the dot product:

$$\mathbf{p} \cdot \mathbf{x} = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = 4(5) + 3(3) + 6(2) = 20 + 9 + 12 = 41$$

The consumer spends \$41. The same expression gives the budget constraint: with income m , the bundles the consumer can just afford are those that satisfy

$$\mathbf{p} \cdot \mathbf{x} = m$$

Matrix Multiplication

The rule for multiplying matrices is designed so that a whole system of equations can be written as a single product. Take the system

$$\begin{aligned} 2x_1 + 3x_2 &= 8 \\ x_1 - x_2 &= 1 \end{aligned}$$

Collect the coefficients in a matrix A and the unknowns in a column vector $\mathbf{x}$. The dot product of row 1 of A with $\mathbf{x}$ gives the left side of the first equation, and the dot product of row 2 with $\mathbf{x}$ gives the left side of the second:

$$A\mathbf{x} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix}$$

Setting this equal to the vector of right-hand sides writes the whole system as one equation:

$$A\mathbf{x} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

The next note solves systems written this way. The general rule repeats the same row-times-column step for every row of the first matrix and every column of the second.

Definition. The product AB is defined only when the number of *columns* of A equals the number of *rows* of B . If A is $m \times n$ and B is $n \times q$, then AB is $m \times q$, and its (i, j) entry is:

$$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

In words: the (i, j) entry of AB is the dot product of row i of A with column j of B .

The order of multiplication matters: in general $AB \neq BA$. Often only one of the two products is even defined.

Example: Two firms each use two inputs, capital (K) and labor (L). The units of each input needed to produce one unit of output are:

	Capital (K)	Labor (L)
Firm 1	3	2
Firm 2	1	4

If capital costs \$10 per unit and labor costs \$8 per unit, what is each firm's cost per unit of output?

We write the input requirements as a matrix R , with one row per firm, and the input prices as a vector $\mathbf{p}$:

$$R = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$$

The vector of unit costs is:

$$R\mathbf{p} = \begin{pmatrix} 3(10) + 2(8) \\ 1(10) + 4(8) \end{pmatrix} = \begin{pmatrix} 46 \\ 42 \end{pmatrix}$$

Firm 1's cost per unit of output is \$46 and Firm 2's is \$42. Each entry of $R\mathbf{p}$ is the dot product of one firm's row of input requirements with the price vector.

Practice Problems

1. Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$.

(a) Compute $A + B$ and $2A - B$.

(b) Compute AB and BA . Are they equal? What does this illustrate?

(c) Verify that $AI_2 = A$ (where I_2 is the 2×2 identity matrix).

2. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

- (a) Write the dimension of each matrix and vector.
 - (b) For each of the following, state whether it is defined. If it is, give the dimension of the result: $A + B$, AB , BA , CA , AC , $A\mathbf{x}$, $\mathbf{x}A$.
 - (c) Compute AB and BA . What does the comparison illustrate?
3. The matrices below record exports among three countries (billions of dollars) in two consecutive years. Rows are exporters and columns are importers.

$$T_{2024} = \begin{pmatrix} 0 & 12 & 8 \\ 5 & 0 & 15 \\ 10 & 7 & 0 \end{pmatrix}, \quad T_{2025} = \begin{pmatrix} 0 & 14 & 6 \\ 6 & 0 & 18 \\ 9 & 9 & 0 \end{pmatrix}$$

- (a) What does the entry in row 3, column 2 of T_{2025} represent? What is its value?
 - (b) Compute $T_{2025} - T_{2024}$. Which bilateral flow grew the most? Which flows fell?
 - (c) Find each country's total exports and total imports in 2025.
 - (d) Let $\mathbf{1}$ be the 3×1 column vector of ones. Compute $T_{2025}\mathbf{1}$ and compare it with your answer to (c).
 - (e) The 2025 flows are converted to euros at 0.92 euros per dollar. Write the converted matrix.
4. Three households buy bread, milk, and coffee. The matrix Q records weekly quantities, with one row per household and one column per good. The vector $\mathbf{p}$ gives prices in dollars.

$$Q = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & 0 \\ 6 & 4 & 2 \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix}$$

- (a) Is $Q\mathbf{p}$ defined? What is its dimension?
 - (b) Compute $Q\mathbf{p}$ and interpret each entry.
 - (c) All prices rise by 5%. Write the new price vector as a scalar multiple of $\mathbf{p}$ and compute each household's new weekly spending. How does it compare with (b)?
 - (d) Instead, only the price of coffee rises, to \$10. Compute each household's new spending. Which household is hit hardest, and why?
5. A furniture maker produces chairs and tables in two regions, North and South. Each chair requires 1 hour of labor and 4 units of wood; each table requires 5 hours of labor and 2 units of wood. Labor costs \$20 per hour in the North and \$12 in the South; wood costs \$3 per unit in the North and \$6 in the South.
- (a) Write the input-requirements matrix A , with one row per product and one column per input, and the input-price matrix W , with one row per input and one column per region.

- (b) Compute AW . What does the entry in row 2, column 1 represent?
 - (c) In which region is each product cheaper to make? Explain the pattern using the input requirements.
 - (d) The product WA is also defined. Does it have an economic meaning? Explain by checking what each entry adds up.
6. A three-good economy has three industries. To produce \$1 of output:
- Industry 1 uses \$0.2 from itself, \$0.3 from Industry 2, and \$0.1 from Industry 3.
 - Industry 2 uses \$0.1 from Industry 1, \$0.2 from itself, and \$0.4 from Industry 3.
 - Industry 3 uses \$0.3 from Industry 1, \$0.1 from Industry 2, and \$0.2 from itself.

Unlike the other problems in this note, each *column* of the matrix belongs to one industry: the entry a_{ij} is the dollar amount of good i used to produce \$1 of good j .

- (a) Write the input-requirements matrix A .
- (b) What does the sum of column 1 represent? Why should it be less than 1?