

6. SECOND DERIVATIVES, CONCAVITY, AND CONVEXITY

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EPH 1100: Analytical Tools

Reading: Hoy et al., §5.5.

- **§5.5:** Higher-order derivatives; concavity and convexity (Definitions 5.7–5.10, Examples 5.31–5.34).
- The risk-preference application (Example 5.35) is optional but worth reading.

1. Why the Second Derivative?

The first derivative tells you whether a function is rising or falling. The second derivative tells you whether it is rising or falling *faster or slower*. In economics, this distinction is often more important than the level or even the direction of change.

Three central economic ideas rest on the sign of the second derivative:

- **Diminishing marginal product:** adding more workers increases output, but each additional worker adds less than the last. The production function is increasing (first derivative positive) but its slope is falling (second derivative negative).
- **Changing marginal cost:** as output rises, the cost of producing an additional unit may increase or decrease. When inputs become scarcer or workers start getting in each other's way, marginal cost rises (second derivative positive). When specialization improves with scale, marginal cost can fall (second derivative negative). The sign of $C''(q)$ tells you which regime the firm is in.
- **Diminishing marginal utility:** each additional unit of a good adds less satisfaction than the one before. The first slice of pizza is wonderful; the fifth is fine; the tenth is a burden. The utility function is increasing (more pizza is still better than less) but its slope is falling (second derivative negative).

In each case, the economic content is in the *curvature* of the function, which is exactly what the second derivative measures.

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2. Second Derivatives: Definition and Notation

The derivative $f'(x)$ is itself a function. If it is differentiable, we can differentiate it again.

Definition. The **second derivative** of f , written $f''(x)$ or $\frac{d^2y}{dx^2}$, is the derivative of the first derivative:

$$f''(x) = \frac{d}{dx} [f'(x)] = \frac{d^2y}{dx^2}$$

If $f''(x)$ exists, we say f is **twice differentiable** at x .

All of the rules you just learned apply: just differentiate once to get $f'(x)$, then differentiate again to get $f''(x)$.

Examples:

$$\begin{aligned} f(x) = x^4 &\implies f'(x) = 4x^3 \implies f''(x) = 12x^2 \\ f(x) = x^{1/2} &\implies f'(x) = \frac{1}{2}x^{-1/2} \implies f''(x) = -\frac{1}{4}x^{-3/2} \\ f(x) = 3x^3 - 12x^2 + 15x &\implies f'(x) = 9x^2 - 24x + 15 \implies f''(x) = 18x - 24 \end{aligned}$$

The third example shows that the second derivative can change sign: $f''(x) > 0$ when $x > 4/3$ and $f''(x) < 0$ when $x < 4/3$.

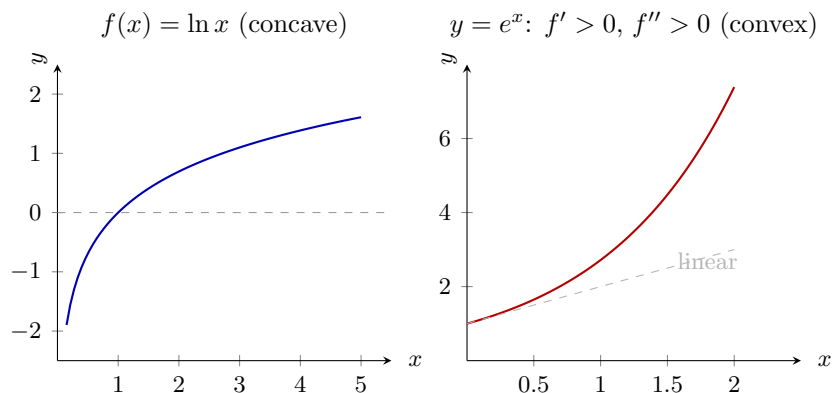
3. Concavity and Convexity

The sign of f'' classifies the *shape* of the function.

Definitions.

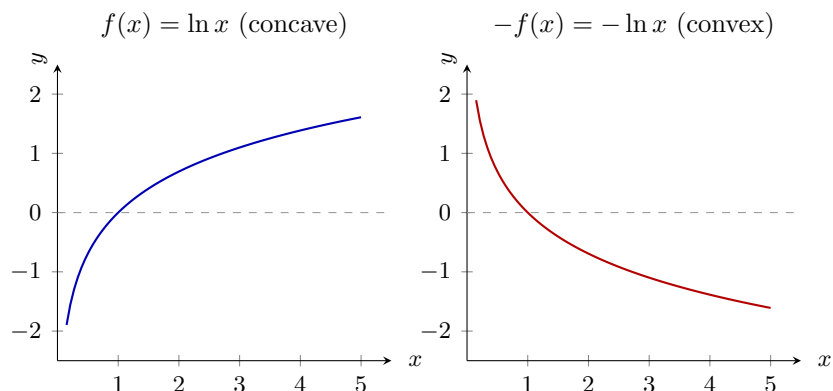
- f is **convex** if $f''(x) \geq 0$ for all x in its domain. (**Strictly convex:** $f''(x) > 0$ except possibly at one point.)
- f is **concave** if $f''(x) \leq 0$ for all x in its domain. (**Strictly concave:** $f''(x) < 0$ except possibly at one point.)

Compare the two functions in Figure 1. Both are increasing (first derivative positive throughout), but they have opposite curvature. The natural log $y = \ln x$ has a slope that is falling: each extra unit of x adds a smaller and smaller increment to y . That is concavity ($f'' < 0$). The exponential $y = e^x$ has a slope that is rising: the function accelerates upward. That is convexity ($f'' > 0$). A linear function ($y = ax + b$) sits exactly between the two: its slope never changes, so $f'' = 0$ everywhere, and it is increasing at a constant rate.

**Figure 1:** Concavity vs. convexity.

Notes: Both functions have $f'(x) > 0$ everywhere shown. The log (left) has $f'' = -1/x^2 < 0$: slope is falling, so each unit adds less than the last (concave). The exponential (right) has $f'' = e^x > 0$: slope is rising, so the function accelerates upward (convex). The dashed line on the right is a linear reference ($y = x + 1$, tangent to e^x at $x = 0$): a linear function has $f'' = 0$ and lies between the two cases.

Note that f is concave if and only if $-f$ is convex. This is a useful symmetry: maximizing a concave function is equivalent to minimizing a convex one. Figure 2 shows this geometrically: flipping a concave hill upside down gives a convex bowl.

**Figure 2:** Flipping a concave function gives a convex function.

Notes: $f(x) = \ln x$ is strictly concave ($f'' < 0$). Its negative, $-f(x) = -\ln x$, is strictly convex ($(-f)'' > 0$). Maximizing $\ln x$ over some set is therefore equivalent to minimizing $-\ln x$ over the same set – a fact used constantly in optimization.

4. Reading a Function from Its Derivatives

Given f' and f'' , you can reconstruct the shape of f without graphing:

$f'(x)$	$f''(x)$	Shape of f
> 0	> 0	Rising, getting steeper (convex, increasing)
> 0	< 0	Rising, getting flatter (concave, increasing)
< 0	> 0	Falling, getting less steep (convex, decreasing)
< 0	< 0	Falling, getting steeper (concave, decreasing)
$= 0$	> 0	At a minimum (slope about to rise)
$= 0$	< 0	At a maximum (slope about to fall)

The last two rows are the second-order conditions for minima and maxima, which we will use in Class 7 to solve optimization problems.

5. Inflection Points

A point where f'' changes sign is called an **inflection point**: the function switches from concave to convex (or vice versa). An inflection point is not a minimum or maximum of f ; it is where the curvature flips.

Example: Let $C(q) = q^3 - 6q^2 + 15q + 10$ (a stylized total cost function).

$$C'(q) = 3q^2 - 12q + 15, \quad C''(q) = 6q - 12$$

Setting $C''(q) = 0$ gives $q = 2$. For $q < 2$, $C'' < 0$ (concave: increasing returns, MC falling). For $q > 2$, $C'' > 0$ (convex: diminishing returns, MC rising). The inflection point at $q = 2$ is where marginal cost reaches its minimum and begins rising (the bottom of the U-shaped MC curve). Figure 3 shows this graphically.

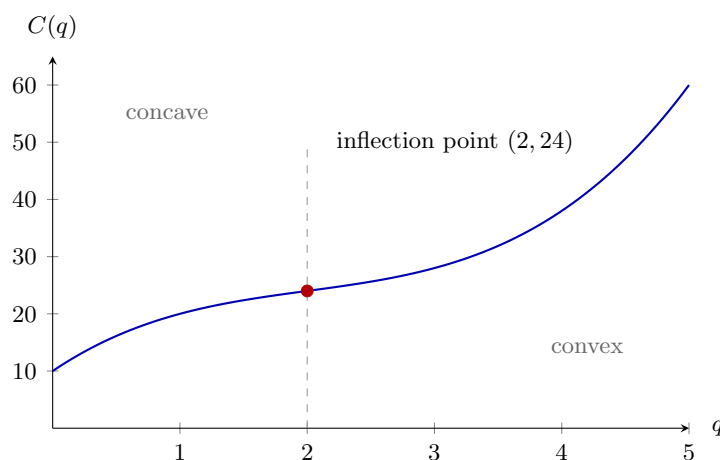


Figure 3: An inflection point.

Notes: $C(q) = q^3 - 6q^2 + 15q + 10$. For $q < 2$, $C''(q) < 0$: the curve bends downward (concave). For $q > 2$, $C''(q) > 0$: the curve bends upward (convex). At $q = 2$, curvature flips – the inflection point.

6. Economic Applications

6.1 Production and the Law of Diminishing Returns

Suppose a firm uses capital K and labor L to produce output y . A simple production function might look like:

$$y = \bar{K} \cdot L$$

where $\bar{K}$ is the fixed stock of machines. In the short run, the firm cannot buy more machines. The only decision is how many workers to hire per machine. Since $\bar{K}$ is a constant, output is a function of L alone, and the **marginal product of labor** – the additional output from one more worker – is:

$$MP(L) = \frac{dy}{dL} = \bar{K}$$

Constant marginal product: every worker is equally productive, regardless of how many are already employed. This is a special case. A more realistic short-run production function is:

$$y = \bar{K} \cdot L^{1/2}$$

Now:

$$MP(L) = \frac{dy}{dL} = \frac{\bar{K}}{2\sqrt{L}}$$

Notice that $MP(L)$ is now a function of L . As you hire more workers, the marginal product falls. This is the **law of diminishing marginal returns**, and the logic behind it is simple: in the short run, you hire the most productive worker first and put them on the most productive task. Then you hire the next best worker for the next best task, and so on. Each additional worker you hire is a little less productive than the last – not because they are worse people, but because the best slots are already filled. In calculus terms:

$$f''(L) = \frac{d^2y}{dL^2} = -\frac{\bar{K}}{4}L^{-3/2} < 0 \quad \text{for all } L > 0$$

A negative second derivative means diminishing marginal returns: the production function is concave.

Contrast this with $y = \bar{K} \cdot L^{3/2}$: here $f''(L) = \frac{3\bar{K}}{4}L^{-1/2} > 0$, so the marginal product is *increasing*. Each additional worker is more productive than the last. This can happen over some range when workers are so few that adding one allows the whole team to specialize further (one person cuts, one assembles, one inspects – everyone benefits). This regime typically reverses once the team grows large enough that the fixed capital becomes the constraint.

6.2 Cost Functions and Curvature

There is a direct relationship between the shape of the production function and the shape of the cost function.

When the production function is concave (diminishing marginal returns), more input is required per unit of output as production rises. At constant input prices, cost per unit of output is therefore rising. The **cost function is convex**.

Example: Production function $y = L^{1/2}$, wage w , fixed cost c_0 .

From $y = L^{1/2}$, we get $L = y^2$. Total cost is $C(y) = c_0 + wy^2$.

$$MC(y) = C'(y) = 2wy \quad (\text{rising in output})$$

$$C''(y) = 2w > 0 \quad (\text{cost function is convex, as expected})$$

The inverse also holds: a convex production function (increasing marginal returns) yields a concave cost function. This mapping between production and cost curvature is used constantly in micro theory.

Practice Problems

1. Find the first and second derivatives of each function. Classify each as concave, convex, or neither on its natural domain.
 - (a) $f(x) = x^3$
 - (b) $f(x) = -x^2 + 4x$
 - (c) $f(x) = x^{2/3}$, $x > 0$
 - (d) $f(x) = x^3 - 6x^2 + 9x$
2. A firm's production function is $y = 4L^{3/4}$.
 - (a) Find the marginal product of labor $MP(L)$.
 - (b) Is the marginal product increasing or decreasing in L ? Show using f'' .
 - (c) If the wage is $w = 2$ and fixed costs are $c_0 = 100$, find the cost function $C(y)$. (Invert the production function to write L as a function of y , then substitute into $C = c_0 + wL$.)
 - (d) Is the cost function concave or convex? Is this consistent with your answer in part (b)?
3. A firm's short-run output depends on its capital stock, $y = f(K)$.
 - (a) For $y = 2K^{1/3}$, find $f''(K)$ and classify the curvature. What does this imply about the marginal product of capital?
 - (b) For $y = 0.5K^2$, find $f''(K)$ and classify the curvature. Contrast your answer with part (a).
 - (c) Which of the two functional forms is more realistic for a firm operating with a fixed amount of other inputs? Why?

4. Consider the cost function $C(q) = \frac{1}{3}q^3 - 2q^2 + 5q + 12$.
- (a) Find $MC(q) = C'(q)$ and $C''(q)$.
 - (b) Find the inflection point of C . What happens to marginal cost at this point?
 - (c) For what range of output is the cost function convex? Concave?
 - (d) Suppose this is a logistics firm, and q is measured in millions of units (say, packages). What would explain the behavior of this cost function?
5. A consumer's utility from consuming x cups of coffee per day is $u(x) = 8x^{1/2}$, $x \geq 0$.
- (a) Find the marginal utility $MU(x) = u'(x)$.
 - (b) Show that $u''(x) < 0$. What does this imply about the marginal utility of each additional cup?
 - (c) Compute $MU(1)$ and $MU(4)$. Does the second cup add more or less satisfaction than the first? Interpret. Is this what economic theory predicts?