

3. LIMITS, CONTINUITY, AND TANGENT LINES

J.P. Bastos*

EPH 1100: Analytical Tools

Reading: Hoy et al., §4.1–4.2, §5.1.

- **§4.1:** Continuity of a function: definition and intuition (skip the epsilon-delta proofs).
- **§4.2:** Economic applications: where discontinuities appear and why they matter.
- **§5.1:** The tangent line and the derivative: secant lines, limits, and slope at a point.

1. Why This Matters

Consider two examples. A candidate who wins 50.1% of the vote in a legislative race is elected; one who wins 49.9% is not. A student who scores 1,200 on the SAT qualifies for a merit scholarship; one who scores 1,199 does not. In both cases, the function relating the input (vote share or test score) to the output (elected or not, scholarship received) jumps discontinuously at a threshold. Figure 1 shows the structure.

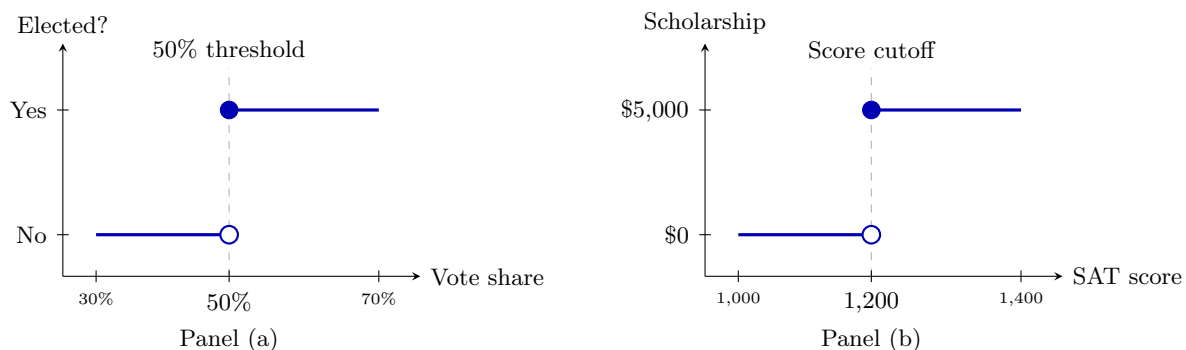


Figure 1: Discontinuous functions at thresholds.

Notes: Panel (a): a candidate winning more than 50% of the vote is elected; one winning less is not. Panel (b): a student scoring at or above 1,200 on the SAT receives a \$5,000 merit scholarship; one scoring below receives nothing. In both cases, the function jumps discontinuously at the threshold even though the inputs just below and above are nearly identical.

Threshold rules like these are common in policy and social science: tax brackets change at income cutoffs, benefit eligibility depends on age, regulatory standards shift at fixed thresholds.

* Assistant Professor of Economics, University of Austin (UTX). jbastos@uaustrin.org

Wherever such a rule appears, the underlying function is discontinuous at the cutoff. This creates an apparent problem: the main tools we build in this course (derivatives, optimization, comparative statics) require functions to respond smoothly to small changes. When a function jumps, those tools break down at the jump.

The concept of a **limit** is the way around this. Even when a function is undefined or jumps at a point, we can ask: what value does it approach as the input gets close to that point from below? From above? Limits let us describe near-threshold behavior rigorously, even at points where the function itself misbehaves. They also underpin the **regression discontinuity design**: by comparing outcomes just below and just above a threshold (where individuals are otherwise nearly identical), researchers can estimate causal effects without a randomized experiment. Section 4 below builds directly on limits to define the derivative; Class 4 then develops systematic rules for computing it.

2. Limits

A **limit** asks: what value does $f(x)$ approach as x gets arbitrarily close to some given number (say -10 , 0.5 , or 327)? Call that number a . We then write:

$$\lim_{x \rightarrow a} f(x) = L$$

The key is that we never require $x = a$, only that x gets closer and closer. The function need not even be defined at a for the limit to exist (see Example 2).

Example 1: Let $f(x) = 3x - 2$. What happens as $x \rightarrow 4$?

As x approaches 4, $f(x)$ approaches $3(4) - 2 = 10$. There is nothing subtle here: $\lim_{x \rightarrow 4} (3x - 2) = 10$. For a linear function, the limit at any point simply equals the function's value there. We will see shortly that this is the definition of continuity.

Example 2: Now consider:

$$f(x) = \frac{x^2 - 4}{x - 2}$$

This is not defined at $x = 2$ (division by zero). But factor the numerator:

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2 \quad \text{for } x \neq 2$$

As $x \rightarrow 2$, the expression approaches 4. So $\lim_{x \rightarrow 2} f(x) = 4$, even though $f(2)$ does not exist. This is why limits matter: they describe what a function approaches, which can be well-defined even where the function itself is not.

Example 3: Consider $f(x) = 1/x$ as $x \rightarrow 0$.

As x approaches 0 from the right, $1/x$ grows without bound: $+\infty$. As x approaches 0 from the left, $1/x$ falls without bound: $-\infty$. The two sides behave completely differently, so $\lim_{x \rightarrow 0} 1/x$ does not exist.

One-sided limits

The example above illustrates a general concept. The **left-hand limit** and **right-hand limit** are:

$$\lim_{x \rightarrow a^-} f(x) \quad (\text{approaching from the left}) \qquad \lim_{x \rightarrow a^+} f(x) \quad (\text{approaching from the right})$$

The two-sided limit $\lim_{x \rightarrow a} f(x)$ exists if and only if both one-sided limits exist and are equal. If they disagree, the limit does not exist.

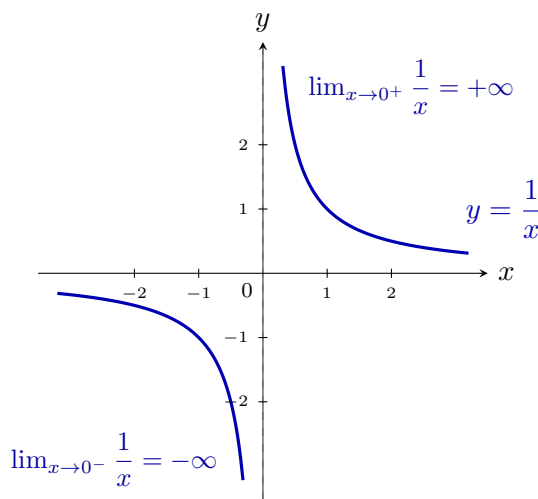


Figure 2: The graph of $y = 1/x$.

Notes: Approaching $x = 0$ from the right, $y \rightarrow +\infty$; from the left, $y \rightarrow -\infty$. The one-sided limits disagree, so the two-sided limit does not exist.

3. Continuity

Intuitively, a function is continuous if you can draw its graph without lifting your pen. The formal definition makes this precise.

Definition. A function f is **continuous at a point** a if:

1. $f(a)$ is defined,
2. $\lim_{x \rightarrow a} f(x)$ exists,
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

Each condition addresses a distinct way continuity can fail, and the three are easy to confuse.

Condition 1 – “ $f(a)$ is defined” – means that plugging $x = a$ into the formula produces a real number. No division by zero, no undefined operation. Example: $f(x) = 1/x$ is undefined at $x = 0$ because $1/0$ is not a number. Condition 1 fails there.

Condition 2 – “the limit exists at a ” – is a statement about the function’s behavior near a , not at a . For the limit to exist, $f(x)$ must approach the same finite value L as x approaches a from both sides. Conditions 1 and 2 are independent: in $f(x) = (x^2 - 4)/(x - 2)$, the limit exists ($\lim_{x \rightarrow 2} f(x) = 4$) but the function is undefined at $x = 2$; in the bonus program below, S is defined at $\bar{h}$ (its value there is $w\bar{h}$) but the one-sided limits disagree, so no two-sided limit exists.

Condition 3 – “the limit equals $f(a)$ ” – is the final requirement: $f(a)$ and $\lim_{x \rightarrow a} f(x)$ must not only both exist, but agree. Imagine taking Example 2’s function and defining $f(2) = 7$ by hand. Then $f(2)$ exists (it’s 7), the limit exists (it’s 4), but they disagree. The function jumps to the wrong value at that point. Condition 3 rules this out.

The clearest way to see these distinctions in practice is through the ways continuity can fail.

Infinite discontinuity

The function $f(x) = 1/x$ fails both of the first two conditions at $x = 0$. Condition 1 fails: the function is undefined at $x = 0$ (division by zero). Condition 2 also fails: the one-sided limits are $+\infty$ and $-\infty$, which disagree, so the two-sided limit does not exist. Both failures happen simultaneously here, but they are separate: either one alone is enough to break continuity. Economically, this shape arises naturally: the rectangular hyperbola $y = \alpha/x$ describes demand implied by Cobb-Douglas utility, where consumers always spend the same total amount regardless of price. The function is perfectly well-behaved everywhere except at $x = 0$, which is outside the economically relevant domain.

Jump discontinuity

The left- and right-hand limits both exist, but they are not equal. The function jumps at a .

The most important source of jump discontinuities in economics is discreteness. Real inputs are often not perfectly divisible: you cannot hire 1.5 workers or install half a machine.¹ Policy thresholds create the same structure. A bonus program paying B dollars if hours worked exceed $\bar{h}$ generates a jump at $\bar{h}$:

$$S(h) = \begin{cases} wh & \text{if } h \leq \bar{h} \\ wh + B & \text{if } h > \bar{h} \end{cases}$$

The left-hand limit at $\bar{h}$ is $w\bar{h}$; the right-hand limit is $w\bar{h} + B$. They differ by $B > 0$, so the function jumps.²

¹When inputs are discrete, the production function $f(L)$ is technically defined only on integers $L \in \{0, 1, 2, \dots\}$, making it discontinuous between integer values. Continuous approximations are used for tractability, but discreteness reappears as a constraint in problems that require integer solutions.

²A worker just short of $\bar{h}$ has a large marginal payoff from one more hour (capturing the full bonus B); a worker just past $\bar{h}$ earns only the regular wage. Standard prediction: workers bunch at exactly $\bar{h}$ to capture the bonus, then

Operations that preserve continuity

As a technical note, I must tell you that sums, differences, products, and (non-zero) quotients of continuous functions are continuous. The same applies to compositions, so any function built from polynomials and exponentials is continuous everywhere on its natural domain. In other words, as a practical note, unless it is quite obvious that the function is discontinuous (think of a function with an explicit if threshold), basically every function in economics will be continuous by design (theorists have their clever tricks!), and pretty much any operation you perform on them will not change that fact.

4. The Tangent Line

Suppose a firm's total cost is $C(q) = 10q$: each unit costs exactly \$10, regardless of how many you have already made. Produce one more unit from any output level, and the cost rises by \$10. The rate of change is constant everywhere, and measuring it at any single point is trivial.

Now suppose the cost function is $C(q) = q^2$. Total cost at $q = 10$ is \$100; at $q = 20$, it is \$400. You can ask: what is the average increase in cost per unit over this range?

$$\frac{C(20) - C(10)}{20 - 10} = \frac{400 - 100}{10} = 30$$

The average is \$30 per unit, and that's exactly what "rise over run" would tell you. But the average of 30 cannot tell you what is happening at any specific point.³ The economically meaningful question is narrower: how much does one more unit cost right at $q = 10$? "Rise over run" cannot answer the question at exactly that point, even though this is what marginal cost means in the strict sense.

Shrinking the interval

The approach is to compute the average rate of change over smaller and smaller intervals starting from $q = 10$, and ask what value that average approaches.

Example: Let $C(q) = q^2$. For each $q > 10$, compute the average cost increase per unit between $q_0 = 10$ and $q = q$:

stop. Economists exploit these sharp thresholds in empirical work: a **regression discontinuity** design uses the jump to identify causal effects, comparing individuals just below and just above the cutoff on the assumption that they are otherwise similar.

³For instance, we can conceptualize a case where it barely rises from $q = 10$ to $q = 15$, then spikes sharply from $q = 15$ to $q = 20$.

q	Δq	$C(q) = q^2$	$\frac{C(q) - 100}{\Delta q}$
11	1	121	21.0
10.5	0.5	110.25	20.5
10.1	0.1	102.01	20.1
10.01	0.01	100.2001	20.01

As the interval shrinks, the average converges to 20. To confirm this algebraically, expand for any $\Delta q \neq 0$:

$$\frac{C(10 + \Delta q) - C(10)}{\Delta q} = \frac{(10 + \Delta q)^2 - 100}{\Delta q} = \frac{100 + 20\Delta q + (\Delta q)^2 - 100}{\Delta q} = 20 + \Delta q$$

As $\Delta q \rightarrow 0$, this approaches exactly 20. The marginal cost at $q = 10$ is \$20 per unit.

Geometrically, each Δq you are computing the slope of the line connecting the points $(q, C(q))$ and $(q + \Delta q, C(q + \Delta q))$ on the curve. As Δq shrinks, this line approaches a fixed limiting position: the **tangent line** at the point, whose slope is the instantaneous rate of change.

Definition. The slope of the **tangent line** to f at x is:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This limit, when it exists, is the **derivative** of f at x .

For $C(q) = q^2$, the derivative at $q = 10$ is $C'(10) = 20$. This is the instantaneous marginal cost at that output level. If you ask the same question at $q = 5$, the same algebra gives $\frac{(5 + \Delta q)^2 - 25}{\Delta q} = 10 + \Delta q \rightarrow 10$, confirming $C'(5) = 10$. Marginal cost is rising as output rises, which is why the curved case is more interesting – and more realistic – than the average over a segment implied by “rise over run”. The next class develops systematic rules for computing these derivatives without going through a table each time.

Practice Problems

1. Evaluate each limit, or explain why it does not exist.

(a) $\lim_{x \rightarrow 5} (4x - 3)$

(b) $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

(c) $\lim_{x \rightarrow 0^+} \frac{1}{x}$ and $\lim_{x \rightarrow 0^-} \frac{1}{x}$. Does $\lim_{x \rightarrow 0} \frac{1}{x}$ exist?

2. A government benefit pays \$800 per month to households with income below \$24,000 per year, and nothing above that threshold:

$$B(Y) = \begin{cases} 800 & \text{if } Y < 24,000 \\ 0 & \text{if } Y \geq 24,000 \end{cases}$$

- (a) Compute $\lim_{Y \rightarrow 24000^-} B(Y)$ and $\lim_{Y \rightarrow 24000^+} B(Y)$.
- (b) Is B continuous at $Y = 24,000$? Apply the three-part definition.
- (c) What happens to a household earning \$23,999 if it gets a \$2 raise? Describe the incentive this creates.
- (d) Policymakers call designs like B a *benefit cliff*, and often replace them with a gradual phase-out instead:

$$\tilde{B}(Y) = \begin{cases} 800 & \text{if } Y < 16,000 \\ 800 - 0.1(Y - 16,000) & \text{if } 16,000 \leq Y < 24,000 \\ 0 & \text{if } Y \geq 24,000 \end{cases}$$

Check whether $\tilde{B}$ is continuous at $Y = 16,000$ and at $Y = 24,000$. Why might a policymaker prefer this design to the original B ?

3. Let $C(q) = q^2 + 4$ be a firm's total cost function and let $P = (3, 13)$.
- (a) Compute the average rate of change of C from $q = 3$ to $q = 3 + \Delta q$ for $\Delta q = 1, 0.5, 0.1$.
 - (b) Take the limit as $\Delta q \rightarrow 0$ to find the marginal cost at $q = 3$. (As in the example on pp. 5-6)