

2. FUNCTIONS AND ECONOMIC RELATIONSHIPS

J.P. Bastos*

EPH 1100: Analytical Tools

Reading: Hoy et al., §2.4.

- **§2.4:** Functions: definition, domain and range, linear, quadratic, power, and Cobb-Douglas functions. Read alongside these notes.

1. Functions as Economic Relationships

Almost every statement in economics is a statement about how one thing depends on another. The quantity of coffee people buy depends on its price. A firm's output depends on how much labor it hires. A worker's income depends on the wage and the hours worked. In each case, one variable is determined by another, according to some rule.

A **function** is a precise way of writing down that rule. Formally: given two sets X and Y , a function $f : X \rightarrow Y$ assigns to each element of X exactly one element of Y . In economics, X and Y are almost always sets of real numbers, so we write $y = f(x)$.

Three vocabulary terms to keep in mind:

- **Domain:** the set of values x is allowed to take. In economics, the domain is often restricted by reality: prices cannot be negative, quantities must be non-negative.
- **Range:** the set of values $f(x)$ actually takes as x varies over the domain.
- **Rule:** the formula relating x to y .

The “exactly one element” requirement matters: for each price, there is one quantity demanded, not two, not zero. A relationship that assigns multiple outputs to the same input is not a function.

Why the domain matters

Consider the demand function $Q^D = 100 - 2P$. Mathematically, this is defined for all real P . But economically, we need $P \geq 0$ (prices are not negative) and $Q^D \geq 0$ (quantities are not negative). The second condition gives $100 - 2P \geq 0$, so $P \leq 50$. The economically relevant domain is therefore $P \in [0, 50]$.

*Assistant Professor of Economics, University of Austin (UATX). jbastos@uaustrin.org

2. Linear Functions

The most important function type in this course is the linear function:

$$y = a + bx$$

where a is the **intercept** (the value of y when $x = 0$) and b is the **slope**.

The slope is the key concept here. It tells you the rate of change: how much y changes for each one-unit increase in x .

$$b = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

If $b > 0$, the function is increasing: higher x means higher y . If $b < 0$, it is decreasing. If $b = 0$, y does not depend on x at all.

Example: A market's demand curve is:

$$Q^D = 100 - 2P$$

The slope is -2 : for every \$1 increase in price, quantity demanded falls by 2 units. The intercept is 100: if the price were zero, consumers would demand 100 units.

Question: What is the demand when $P = 20$? When $P = 30$?

$$Q^D(20) = 100 - 2(20) = 60$$

$$Q^D(30) = 100 - 2(30) = 40$$

A \$10 increase in price reduces quantity demanded by 20 units. That is exactly the slope (-2 per dollar $\times$ \$10 = -20 units), which is what makes the slope so useful.

The Marshall convention

Economists almost always graph demand with *price on the vertical axis* and *quantity on the horizontal axis* (the reverse of what you might expect from the equation $Q^D = 100 - 2P$, which has Q as the dependent variable). This is a historical convention from Alfred Marshall (1890) and is used throughout economics.

To graph the demand curve, it helps to find two points:

- **Vertical intercept** (set $Q = 0$): $0 = 100 - 2P \implies P = 50$.
- **Horizontal intercept** (set $P = 0$): $Q = 100$.

The line connecting $(Q, P) = (0, 50)$ and $(100, 0)$ is the demand curve.

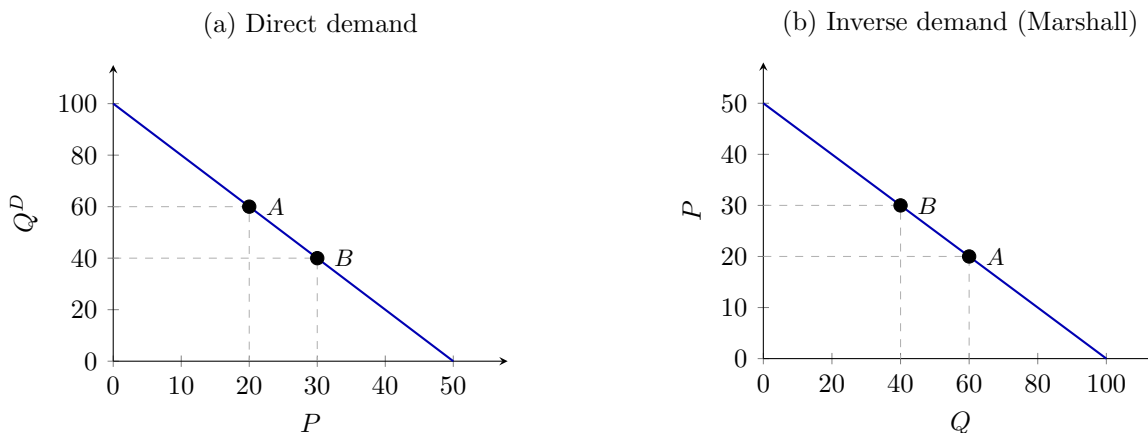


Figure 1: Demand and inverse demand.

Notes: Panel (a) plots $Q^D = 100 - 2P$ directly: price on the horizontal axis, quantity on the vertical – the natural way to graph a function of P . Panel (b) rearranges to the **inverse demand** $P = 50 - \frac{1}{2}Q$ and graphs with Q on the horizontal axis, following the Marshall convention. Both panels represent the same relationship; only the roles of the axes are swapped. Points A ($P = 20$, $Q = 60$) and B ($P = 30$, $Q = 40$) correspond to the calculations in the worked example above.

Monotonicity

Every function must assign exactly one output to each input, as required by the definition in Section 1. Panel (b) of Figure 1 takes $Q^D = 100 - 2P$, which assigns one quantity to each price, and rearranges it into $P = 50 - \frac{1}{2}Q$, treating P as if it were now a function of Q . That step needs justification. Rearranging an equation algebraically is always valid for any single number you plug in, but it says nothing, on its own, about whether the rearranged expression behaves like a function across the whole domain. It could easily turn out that some quantity Q corresponds to two different prices, in which case $P = 50 - \frac{1}{2}Q$ would not be a function at all. The property that rules this out is called monotonicity.

Definition. A function f is **monotonically increasing** on an interval if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$ (**strictly increasing** if the inequality is strict), and **monotonically decreasing** if $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$. The function moves in one direction only, never reversing course.

In plain terms: a monotonic function never reverses direction. An increasing function may keep rising, or level off for a stretch, but it can never turn around and start falling; a decreasing function works the same way in reverse. Invertibility needs something slightly stronger than this: **strict** monotonicity, where the function is never flat either, so that no two different values of x ever produce the same y . To invert f means to read it backward: given an output y , recover the one input x that produced it. If f is strictly monotonic, this always works, because each value of y is produced by exactly one x as x sweeps across the domain. If f is not strictly monotonic, either because it reverses direction or because it is flat somewhere, some value of y is produced by more than one x , and reading backward from that y gives more than one candidate answer. That breaks

the definition of a function applied to the inverse: one output for every input. Figure 2 shows exactly this failure.

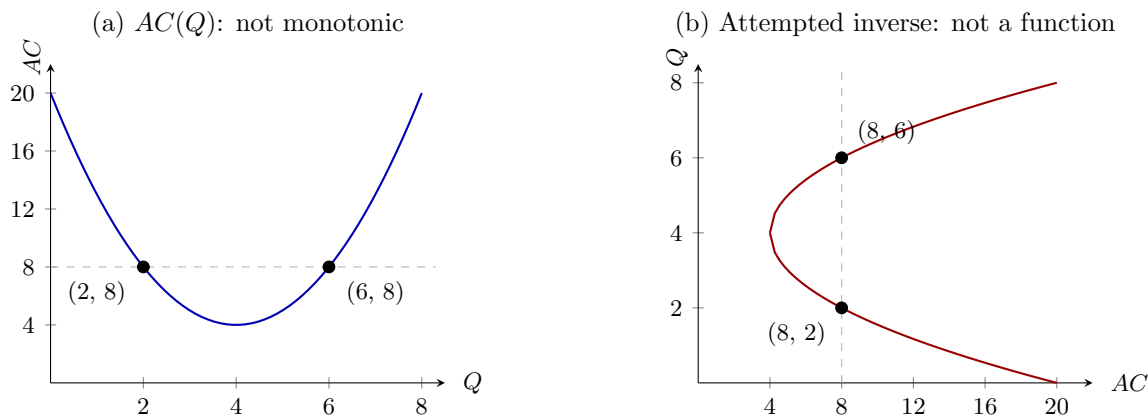


Figure 2: Inverting a non-monotonic function fails.

Notes: Panel (a) plots $AC(Q) = Q^2 - 8Q + 20$: the curve falls, then rises, so it is not monotonic. Both $Q = 2$ and $Q = 6$ give exactly $AC = 8$ (dashed line), so the same output is produced by two different inputs. Panel (b) attempts the inverse: swap the axes and plot Q against AC , exactly as Figure 1 does for demand. The result fails the vertical line test. At $AC = 8$, the vertical dashed line crosses the plotted relation twice, at $Q = 2$ and at $Q = 6$. There is no single value to assign to Q when $AC = 8$, so this is not a function of AC at all. Contrast this with Figure 1, where demand is monotonic and the swap in panel (b) produces a genuine function.

Any linear function with $b \neq 0$ is strictly monotonic on its whole domain, since the slope never changes sign. That is why solving $Q^D = 100 - 2P$ for P is valid: demand is strictly decreasing, so every Q corresponds to a unique P , and $P = 50 - \frac{1}{2}Q$ is a genuine function of Q .

Not every function is monotonic. Figure 2 is an example: the quadratic average cost curve there falls, then rises, so it cannot be inverted over its whole domain, only after restricting to one branch where the direction never reverses (e.g., $Q \geq 4$, where the curve is monotonically increasing). Think of this way: we can answer the question “at which quantities the average cost is equal to X ?”, but because there are multiple quantities, when we invert, we cannot find the AC corresponding to that quantity.

3. Shifts vs. Movements Along a Curve

A demand function has the form $Q^D = f(P; \text{other variables})$. Price P is the variable on the axis; the “other variables” (income, prices of related goods, tastes) are the parameters that shift the curve.

- **Movement along the curve:** a change in P (the variable on the axis). You move from one point to another on the same curve. Example: price rises from \$20 to \$30, quantity demanded falls from 60 to 40.
- **Shift of the curve:** a change in a parameter. The entire curve moves. Example: consumer income rises, so at every price, consumers demand more. The new curve is $Q^D = 120 - 2P$,

shifted outward.

This is a direct application of the endogenous/exogenous distinction from Class 1. Within the demand model, price is the variable we trace along the curve (endogenous, determined in the market). Income is a parameter: it shifts the curve when it changes (exogenous to the demand model itself).

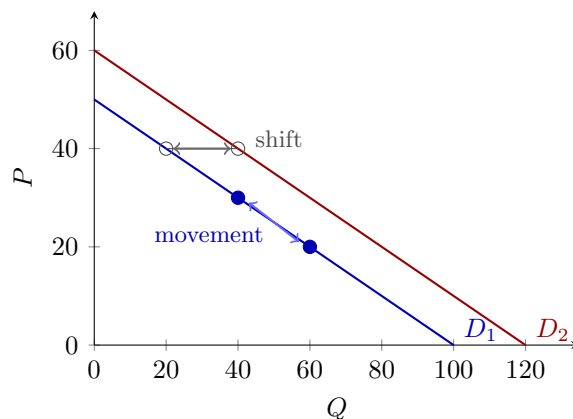


Figure 3: Movement vs. shift.

Notes: The blue arrow shows a **movement along** D_1 : price rises from 20 to 30, quantity falls from 60 to 40 — same curve, different point. The black arrow shows a **shift**: income rises, moving the entire curve rightward from D_1 to D_2 at every price level.

4. Other Function Types in Economics

Linear functions are the workhorse, but economics uses several other shapes.

Quadratic functions

$$y = ax^2 + bx + c, \quad \text{where } a \neq 0.$$

- If $a > 0$: U-shaped (convex). Typical shape of an average cost curve — the curve falls initially, then rises as the firm hits capacity.
- If $a < 0$: inverted-U (concave). Typical shape of a profit curve — profit rises, peaks, then falls.

The minimum (if $a > 0$) or maximum (if $a < 0$) occurs at $x^* = -b/2a$. We will derive this properly using derivatives in Class 7.

Power functions

$$y = Ax^b, \quad A > 0.$$

The exponent b controls the shape: $b > 1$ gives an accelerating increase (convex); $0 < b < 1$ gives a decelerating increase (concave); $b = 1$ is linear; $b < 0$ is decreasing, approaching zero as $x \rightarrow \infty$.

The Cobb-Douglas function is the most important application of power functions in economics. In **production**, the function is:

$$Y = AK^\alpha L^\beta$$

where K is capital, L is labor, Y is output, and $A > 0$ is total factor productivity. The exponents α and β are **output elasticities**: a 1% increase in K raises output by $\alpha\%$, holding L fixed. If $\alpha + \beta = 1$, doubling all inputs exactly doubles output (**constant returns to scale**). The same functional form also appears in consumer theory — we return to that application in the optimization unit.

Logarithmic functions

The **natural logarithm** $\ln(x)$ answers the question: to what power must $e \approx 2.718$ be raised to get x ? That is, $y = \ln(x)$ means $e^y = x$. Three rules you will use repeatedly:

- $\ln(xy) = \ln(x) + \ln(y)$ (product $\rightarrow$ sum of logs)
- $\ln(x/y) = \ln(x) - \ln(y)$ (ratio $\rightarrow$ difference of logs)
- $\ln(x^a) = a \ln(x)$ (exponent pulls out as a multiplier)

Two anchor values: $\ln(1) = 0$ (since $e^0 = 1$) and $\ln(e) = 1$ (since $e^1 = e$). The function is defined only for $x > 0$, increases throughout, and is concave — each additional unit of x adds less to $\ln(x)$ than the last.¹

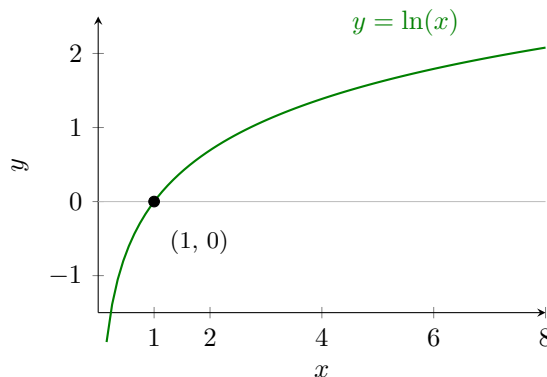


Figure 4: The natural logarithm.

Notes: $\ln(1) = 0$; negative for $0 < x < 1$; positive and increasing for $x > 1$. The concave shape reflects diminishing returns: each additional unit of x adds less to y than the last.

Three reasons economists use logarithms:

¹Why undefined for $x \leq 0$? The natural log is the inverse of e^x . Since $e^x > 0$ for every real x , the equation $e^y = x$ has no real solution when $x \leq 0$.

1. **Cobb-Douglas becomes linear.** Taking logs of $Y = AK^\alpha L^\beta$:

$$\ln Y = \ln A + \alpha \ln K + \beta \ln L$$

This is a linear regression equation. The exponents α and β appear directly as slopes, making them estimable from data using ordinary least squares.

2. **Log-differences are percentage changes.** For small changes, $\Delta \ln(x) \approx \Delta x/x$ — the fractional change in x .

Check. Suppose a country's GDP per capita grows from \$20,000 to \$20,200 — a \$200 increase, or exactly 1%. The log difference is:

$$\ln(20,200) - \ln(20,000) = \ln\left(\frac{20,200}{20,000}\right) = \ln(1.01) \approx 0.00995$$

That is 0.995% — almost exactly the true 1% growth rate. The approximation gets tighter the smaller the change.

This is what makes **log-log regressions** powerful. Estimating $\ln y = \alpha + \beta \ln x + \varepsilon$, the coefficient $\hat{\beta}$ is directly an **elasticity**: a 1% increase in x is associated with a $\hat{\beta}\%$ change in y — no further calculation needed.

Example: Does trade openness raise income? You have data for 100 countries: GDP per capita ranges from about \$1,500 (Haiti) to \$82,000 (Norway). Regressing GDP per capita *in levels* on trade share is problematic. A \$1,000 increase means a 67% rise in Haiti but barely 1% in Norway — the same regression slope cannot meaningfully describe both.

Taking logs resolves this. The slope of $\ln(\text{GDP per capita})$ on $\ln(\text{trade share})$ is an elasticity, directly comparable across all income levels. An estimate of $\hat{\beta} = 0.3$ means: countries with 1% higher trade share tend to have 0.3% higher GDP per capita.

3. **Diminishing marginal utility.** The utility function $U(x) = \ln(x)$ captures the idea that each additional unit of x is worth less than the previous one — a direct consequence of the concavity of the log function. This is the workhorse specification for consumer theory and risk aversion.

5. Algebra Review: Solving and Substituting

Two operations appear constantly in this course.

Solving for an unknown

Given an equation with one unknown, isolate the variable.

Example: Solve $100 - 2P = 60$ for P .

$$100 - 2P = 60 \implies 2P = 40 \implies P = 20$$

Substitution

Replace one variable using another equation.

Example: A market has $Q^D = 100 - 2P$ and $Q^S = 3P - 20$. Find the equilibrium price and quantity.

Step 1: Set $Q^D = Q^S$ (equilibrium condition):

$$100 - 2P = 3P - 20$$

Step 2: Solve for P :

$$120 = 5P \implies P^* = 24$$

Step 3: Substitute back to find Q^* :

$$Q^* = 100 - 2(24) = 52$$

At $P = 24$, buyers want exactly 52 units and sellers want to supply exactly 52 units. The market clears.

Practice Problems

1. A supply function is $Q^S = 6P - 18$.
 - (a) What is the minimum price at which any quantity is supplied?
 - (b) How much is supplied at $P = 10$? At $P = 15$?
 - (c) What is the slope? Interpret it.
2. Using $Q^D = 80 - P$ and $Q^S = 2P - 10$:
 - (a) Find the equilibrium price and quantity.
 - (b) Suppose a new study increases demand to $Q^D = 95 - P$. Find the new equilibrium.
 - (c) Is the change from (a) to (b) a movement along the supply curve or a shift? Explain.
 - (d) Graph $Q^D = 80 - P$ and $Q^S = 2P - 10$ using the Marshall convention, labeling both intercepts and the equilibrium point from (a). Then add the shifted demand curve from (b) to the same graph.
3. A firm's total cost function is $TC = Q^2 - 4Q + 20$.

- (a) What is TC when $Q = 0$? What does this represent?
 - (b) At what output level is TC minimized? (Use the formula $Q^* = -b/2a$.)
 - (c) Write the firm's average cost function.
4. A country's output follows the Cobb-Douglas function $Y = AK^{0.4}L^{0.6}$.
- (a) Take the log of the function: $\ln(Y) = ?$
 - (b) Simplify $\ln(K^3) - \ln(K)$ to a single term.
 - (c) A simple wage equation is $\ln(w) = \ln(20) + 0.5 \ln(S)$, where S is years of schooling. Rewrite this without logarithms, expressing w as a function of S .
5. Solve each of the following for the variable indicated.
- (a) $e^{0.04x} = 3$, for x .
 - (b) A firm's revenue after t years is $R(t) = R_0 e^{gt}$, where R_0 is current revenue and g is the annual growth rate. Revenue doubles after 10 years, so $R(10) = 2R_0$. Solve for g . (Use $\ln(2) \approx 0.693$.)
 - (c) A firm's short-run production function is $Q = 5\sqrt{L}$. How many workers L are needed to produce $Q = 60$?
6. Practice isolating a variable in a general relationship.
- (a) An isocost line is $C = wL + rK$. Solve for K as a function of L (and the parameters w , r , C).
 - (b) A Cobb-Douglas production function is $Y = AK^\alpha L^\beta$. Solve for L as a function of Y , A , K , α , and β .
 - (c) A constant-elasticity demand curve is $\frac{Q}{Q_0} = \left(\frac{P}{P_0}\right)^{-\varepsilon}$, where Q_0 and P_0 are a reference quantity and price and $\varepsilon > 0$ is the elasticity of demand. Solve for P as a function of Q , Q_0 , P_0 , and ε .